THE PULLBACKS OF PRINCIPAL COACTIONS

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Abstract: We prove that the class of principal coactions is closed under one-surjective pullbacks in an appropriate category of algebras equipped with left and right coactions. This allows us to handle cases of C^*-algebras lacking two different non-trivial ideals. As an example, we carry out an index computation for noncommutative line bundles over the standard Podleś sphere using the Mayer-Vietoris type arguments afforded by a one-surjective pullback presentation of the C^*-algebra of this quantum sphere.

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1 Introduction and preliminaries

The idea of decomposing a complicated object into simpler pieces and connecting data is a fundamental computational principle throughout mathematics. In the case of (co)homology theory, it yields the Mayer-Vietoris long exact sequence whose significance and usefulness can hardly be overestimated. The categorical underpinning of all this are pullback diagrams: in a given category they give a rigorous meaning to putting together two objects over a third one.

The goal of this paper is to prove a general pullback theorem for noncommutative Galois theory (principal coactions) and to use the pullback picture of the standard quantum Hopf fibration to compute an index pairing for associated noncommutative line bundles. The former significantly generalizes the main result of [14] that was restricted to comodule algebras and pullbacks of surjections, and the latter provides a new way of computing the aforementioned index pairing for quantum Hopf line bundles (cf. [28]). This pairing was computed in [13] using a noncommutative index formula, and re-derived in [22].

More precisely, our main result is that the pullback of principal coactions over morphisms of which at least one is surjective is again a principal coaction. It may be viewed as a non-linear version of the Bass connecting homomorphism in K-theory [2]. Indeed, linearizing our pullback theorem with the help of a corepresentation of the Hopf algebra yields precisely the Bass construction of a projective module defining the connecting homomorphism in K-theory. On the other hand, our simple example of the standard quantum Hopf fibration shows the need to generalize from two-surjective to one-surjective pullback diagrams, and the pullback method of index computation seems attractive due to its inherent simplicity.

The paper is organized as follows. First, to make our exposition self-contained and to establish notation, we recall fundamental concepts that we use later on. The key Section 2 is devoted to the general pullback theorem for principal coactions of coalgebras on algebras, and the final Section 3 is on deriving the index pairing for quantum Hopf line bundles as a corollary to the pullback presentation of the standard Hopf fibration of $SU_q(2)$.

Throughout the paper, we work with algebras and coalgebras over a field. We employ the Heyneman-Sweedler type notation (with the summation symbol suppressed) for the comultiplication $\Delta(c) = c_{(1)} \otimes c_{(2)} \in C \otimes C$ and for coactions $\Delta_V(v) = v_{(0)} \otimes v_{(1)} \in V \otimes C$, ${}_V\Delta(v) = v_{(-1)} \otimes v_{(0)} \in C \otimes V$. The convolution product of two linear maps from a coalgebra to an algebra is denoted by $*$: $(f * g)(c) := f(c_{(1)})g(c_{(2)})$. The set of natural numbers includes 0, that is, $\mathbb{N} = \{0, 1, 2, \ldots\}$.

1.1 Pullback diagrams and fibre products

The purpose of this section is to collect some elementary facts about fibre products. We consider the category of vector spaces as it will be the ambient category for all our pullback diagrams. Let $\pi_1 : A_1 \to A_{12}$ and $\pi_2 : A_2 \to A_{12}$ be linear maps. The *fibre product* of these maps is defined by

$$A_1 \underset{(\pi_1,\pi_2)}{\times} A_2 := \{(a_1, a_2) \in A_1 \times A_2 \mid \pi_1(a_1) = \pi_2(a_2)\}\,. \tag{1.1}$$

Together with the canonical projections

$$\mathrm{pr}_1 : A_1 \underset{(\pi_1,\pi_2)}{\times} A_2 \longrightarrow A_1, \qquad \mathrm{pr}_2 : A_1 \underset{(\pi_1,\pi_2)}{\times} A_2 \longrightarrow A_2\,, \tag{1.2}$$

it forms a universal construction completing the initially given two linear maps into the following commutative diagram:

$$\begin{array}{ccc} A_1 \underset{(\pi_1,\pi_2)}{\times} A_2 & \xrightarrow{\mathrm{pr}_2} & A_2 \\ {\scriptstyle \mathrm{pr}_1}\big\downarrow & & {\scriptstyle \pi_2}\big\downarrow \\ A_1 & \xrightarrow{\pi_1} & A_{12}\,. \end{array} \tag{1.3}$$

Such universal commuting diagrams are called *pullback diagrams*, and fibre products are often referred to as pullbacks.

Next, if $\pi_1 : A_1 \to A_{12}$ and $\pi_2 : A_2 \to A_{12}$ are morphisms of *-algebras, then the fibre product $A_1 \times_{(\pi_1,\pi_2)} A_2$ is a *-subalgebra of $A_1 \times A_2$. Furthermore, if we consider the pullback diagram (1.3) in the category of (unital) C^*-algebras, then $A_1 \times_{(\pi_1,\pi_2)} A_2$ with its componentwise multiplication and *-structure is a (unital) C^*-algebra. Much the same, if B is an algebra and $\pi_1 : A_1 \to A_{12}$ and $\pi_2 : A_2 \to A_{12}$ are morphisms of left B-modules, then the fibre product $A_1 \times_{(\pi_1,\pi_2)} A_2$ is a left B-module via the componentwise left action $b.(a_1, a_2) = (b.a_1, b.a_2)$.

As explained in detail in [16], the pullback of completions of *-algebras does *not* necessarily coincide with the completion of the pullback of *-algebras. Here we prove a useful criterion for the commuting of pullbacks and completions.

Theorem 1.1. *Let* (1.3) *be a pullback diagram in the category of C^*-algebras, and let B_1, B_2, B_{12} be dense subalgebras of A_1, A_2, A_{12}, respectively. Assume that π_1 and π_2 restrict to morphisms $\pi_1^B : B_1 \to B_{12}$ and $\pi_2^B : B_2 \to B_{12}$. Then, if* $\ker(\pi_1) \cap B_1$ *is dense in* $\ker(\pi_1)$ *and π_1^B is surjective, the *-algebra $B_1 \times_{(\pi_1^B,\pi_2^B)} B_2$ is dense in the C^*-algebra $A_1 \times_{(\pi_1,\pi_2)} A_2$.*

Proof. First note that the surjectivity of π_1^B implies the surjectivity of π_1. Indeed, since B_1 and $\pi_1(B_1) = B_{12}$ are by assumption dense in A_1 and A_{12}, respectively, and $\pi_1(A_1)$ is automatically closed, the C^*-homomorphism $\pi_1 : A_1 \to A_{12}$ must be surjective. Next, given $(a, b) \in A_1 \times_{(\pi_1,\pi_2)} A_2$ and $\epsilon > 0$, let $x \in B_1$ and $y \in B_2$ be such that $||x - a|| < \epsilon/4$ and $||y - b|| < \epsilon/4$. Using the triangle inequality and the fact that C^*-algebra morphisms do not increase the norm, we get

$$||\pi_2(y) - \pi_1(x)|| = ||\pi_2(y - b) - \pi_1(x - a)|| \le ||\pi_2(y - b)|| + ||\pi_1(x - a)|| < \epsilon/2. \tag{1.4}$$

Now, let $z \in B_1$ be a lift of $\pi_2(y) - \pi_1(x) \in B_{12}$, and let $[z]$ denote its class in $\tilde{A}_1 := A_1/\ker(\pi_1)$. Since the induced mapping $\tilde{\pi} : \tilde{A}_1 \to A_{12}$, $\tilde{\pi}([p]) := \pi_1(p)$, is an isomorphism of C^*-algebras and $\ker(\pi_1) \cap B_1$ is dense in $\ker(\pi_1)$, we obtain

$$\epsilon/2 > ||\tilde{\pi}^{-1}(\pi_2(y) - \pi_1(x))|| = ||[z]|| = \inf\{||z + k|| \mid k \in \ker(\pi_1) \cap B_1\}. \tag{1.5}$$

Hence there exists a $z_1 \in [z]$ such that $z_1 \in B_1$ and $||z_1|| < \epsilon/2$. It follows from $\pi_1(z_1) = \pi_2(y) - \pi_1(x)$ that $(x + z_1, y) \in B_1 \times_{(\pi_1,\pi_2)} B_2$. Finally, the inequalities

$$||(x + z_1, y) - (a, b)|| \le ||x - a|| + ||y - b|| + ||z_1|| < \epsilon \tag{1.6}$$

prove that $B_1 \times_{(\pi_1,\pi_2)} B_2$ is dense in $A_1 \times_{(\pi_1,\pi_2)} A_2$. $\square$

1.2 The Bass connecting homomorphism in K-theory

Consider a pullback diagram

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ A_1 \arrow[dr, "\pi_1"'] & & A_2 \arrow[dl, "\pi_2"] \\ & A_{12} & \end{tikzcd} \tag{1.7}$$

in the category of unital algebras, and assume that one of the defining morphisms (here we choose π_1) is surjective. Then there exists a long exact sequence in algebraic K-theory [2]

$$\cdots \longrightarrow \mathrm{K}_1(A_{12}) \xrightarrow{\text{Bass}} \mathrm{K}_0(A) \longrightarrow \mathrm{K}_0(A_1 \oplus A_2) \longrightarrow \mathrm{K}_0(A_{12}). \tag{1.8}$$

The mapping $\mathrm{K}_1(A_{12}) \xrightarrow{\text{Bass}} \mathrm{K}_0(A)$ is obtained as follows. First, given left A_i-modules E_i, $i = 1, 2$, we obtain left A_{12}-modules $\pi_{i*}E_i$ defined by $A_{12} \otimes_{A_i} E_i$. Since A_{12} is unital, there are canonical morphisms $\pi_{i*} : E_i \to \pi_{i*}E_i$, $\pi_{i*}(e) = 1 \otimes_{A_i} e$. The modules E_i and $\pi_{i*}E_i$ can be also considered as left modules over the fibre-product algebra A via the left actions given by $a.e_i = \mathrm{pr}_i(a).e_i$, for $e_i \in E_i$, and $a.f_i = \pi_i(\mathrm{pr}_i(a)).f_i$, for $f_i \in \pi_{i*}E_i$. Assume now that $h : \pi_{1*}E_1 \to \pi_{2*}E_2$ is a morphism of left A_{12}-modules. Then $h \circ \pi_{1*} : E_1 \to \pi_{2*}E_2$ and $\pi_{2*} : E_2 \to \pi_{2*}E_2$ can be lifted to morphisms of left A-modules, and we can consider their pullback diagram in the category of left A-modules:

$$\begin{tikzcd} & E_1 \underset{(h\circ\pi_{1*},\pi_{2*})}{\times} E_2 \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ E_1 \arrow[d, "\pi_{1*}"'] & & E_2 \arrow[d, "\pi_{2*}"] \\ \pi_{1*}E_1 \arrow[rr, "h"'] & & \pi_{2*}E_2. \end{tikzcd} \tag{1.9}$$

In [21, Section 2], it is proven in detail that, if E_i is a finitely generated projective module over A_i, $i = 1, 2$, and h is an isomorphism, then the fibre-product $M := E_1 \times_{(h\circ\pi_{1*},\pi_{2*})} E_2$ is a finitely generated A-module. Furthermore, up to isomorphism, every finitely generated projective module over A has this form, and the A_i-modules E_i and $\mathrm{pr}_{i*}M = A_i \otimes_A M$, $i = 1, 2$, are naturally isomorphic. In particular, if E_1 and E_2 are finitely generated free modules, the isomorphism $h : \pi_{1*}E_1 \to \pi_{2*}E_2$ is given by an invertible matrix $U \in \mathrm{GL}_n(A_{12})$. Using the canonical embedding $\mathrm{GL}_n(A_{12}) \subseteq \mathrm{GL}_\infty(A_{12})$, we get a map

$$\mathrm{GL}_\infty(A_{12}) \ni U \longmapsto M \in \mathrm{Proj}(A) \tag{1.10}$$

given by the pullback diagram

$$\begin{tikzcd} & M \arrow[dl] \arrow[dr] & \\ A_1^n \arrow[dr, "\pi_1"'] & & A_2^n \arrow[dl, "\pi_2"] \\ & A_{12}^n \overset{U}{\cong} A_{12}^n\ . & \end{tikzcd} \tag{1.11}$$

This map induces the Bass connecting homomorphism on the level of K-theory. Its explicit description can be found, e.g., in [12]. It is as follows. Assume that $\pi_1 : A_1 \to A_{12}$ is surjective. Then there exist liftings $c, d \in \mathrm{Mat}_n(A_1)$ such that $\pi_1(c) = U^{-1}$ and $\pi_1(d) = U$. Applying [12, Theorem 3.2] to our situation yields $E_1 \times_{(h\circ\pi_{1*},\, \pi_{2*})} E_2 \cong A^{2n}p$, where

$$p = \begin{pmatrix} (c(2-dc)d, 1) & (c(2-dc)(1-dc), 0) \\ ((1-dc)d, 0) & ((1-dc)^2, 0) \end{pmatrix} \in \mathrm{Mat}_{2n}(A). \tag{1.12}$$

Finally, let us mention that it can be argued that the Bass connecting homomorphism exists also for the K-theory of C^*-algebras [11], and is given by the same explicit construction (1.10)–(1.12). Now, due to the Bott periodicity, we obtain the Mayer-Vietoris 6-term exact sequence [4, 26]

$$\begin{tikzcd} \mathrm{K}_0(A) \arrow[r, "(\mathrm{pr}_{1*},\mathrm{pr}_{2*})"] & \mathrm{K}_0(A_1) \oplus \mathrm{K}_0(A_2) \arrow[r, "\pi_{2*}-\pi_{1*}"] & \mathrm{K}_0(A_{12}) \arrow[d, "\partial"] \\ \mathrm{K}_1(A_{12}) \arrow[u, "\mathrm{Bass}"] & \mathrm{K}_1(A_1) \oplus \mathrm{K}_1(A_2) \arrow[l, "\pi_{2*}-\pi_{1*}"'] & \mathrm{K}_1(A) \arrow[l, "(\mathrm{pr}_{1*},\mathrm{pr}_{2*})"'] \end{tikzcd} . \tag{1.13}$$

1.3 Principal extensions and associated projective modules

Recall first the general definition of an entwining structure. Let C be a coalgebra with comultiplication Δ and counit ε, and let A be an algebra with a multiplication m and the unit η. A linear map

$$\psi : C \otimes A \longrightarrow A \otimes C \tag{1.14}$$

is called an *entwining structure* if it is unital, counital, and distributive with respect to both the multiplication and comultiplication:

$$\psi \circ (\mathrm{id} \otimes m) = (m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \psi) \circ (\psi \otimes \mathrm{id}), \qquad \psi \circ (\mathrm{id} \otimes \eta) = (\eta \otimes \mathrm{id}) \circ \mathrm{flip}, \tag{1.15}$$

$$(\mathrm{id} \otimes \Delta) \circ \psi = (\psi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \psi) \circ (\Delta \otimes \mathrm{id}), \qquad (\mathrm{id} \otimes \varepsilon) \circ \psi = \mathrm{flip} \circ (\varepsilon \otimes \mathrm{id}). \tag{1.16}$$

If ψ is an entwining of a coalgebra C and an algebra A, and M is a right C-comodule and a right A-module, we call M an *entwined module* [7] if it satisfies the compatibility condition

$$(ma)_{(0)} \otimes (ma)_{(1)} = m_{(0)}\psi(m_{(1)} \otimes a). \tag{1.17}$$

Next, let P be an algebra equipped with a coaction $\Delta_P : P \to P \otimes C$ of a coalgebra C. Define the coaction-invariant subalgebra of P by

$$B := P^{\mathrm{co}C} := \{b \in P \mid \Delta_P(bp) = b\Delta_P(p),\ \forall p \in P\}. \tag{1.18}$$

We call the inclusion $B \subseteq P$ a C-extension. We call it a *coalgebra-Galois C-extension* if the canonical left P-module right C-comodule map

$$\mathsf{can} : P \underset{B}{\otimes} P \longrightarrow P \otimes C, \quad p \underset{B}{\otimes} p' \longmapsto p\Delta_P(p'), \tag{1.19}$$

is bijective [8]. Note that the bijectivity of can allows us to define the so-called translation map

$$\tau : C \longrightarrow P \underset{B}{\otimes} P, \quad \tau(c) := \mathsf{can}^{-1}(1 \otimes c). \tag{1.20}$$

Moreover, every coalgebra-Galois C-extension comes naturally equipped with a unique entwining structure that makes P a (P,C)-entwined module in the sense of (1.17). It is called the canonical entwining structure [8], and is very useful in calculations or further constructions. Explicitly, it can be written as:

$$\psi(c \otimes p) = \mathsf{can}(\mathsf{can}^{-1}(1 \otimes c)p). \tag{1.21}$$

An algebra P with a right C-coaction Δ_P is said to be *e-coaugmented* if there exists a grouplike element $e \in C$ such that $\Delta_P(1) = 1 \otimes e$. We call the C-extension $B := P^{\mathrm{co}C} \subseteq P$ e-coaugmented. (Much the same way, one defines the coaugmentation of left coactions.) For the e-coaugmented coalgebra-Galois C-extensions, one can show that the coaction-invariant subalgebra defined in (1.18) can be expressed as

$$P^{\mathrm{co}C} = \{p \in P \mid \Delta_P(p) = p \otimes e\}. \tag{1.22}$$

Indeed, Formula (1.21) allows us to express the right coaction in terms of the entwining:

$$\Delta_P(p) = \psi(e \otimes p), \tag{1.23}$$

and Equation (1.15) yields the right-in-left inclusion. The opposite inclusion is obvious.

Next, if ψ is invertible, one can use (1.16) to show that the formula

$${}_P\Delta(p) := \psi^{-1}(p \otimes e) \tag{1.24}$$

defines a left coaction ${}_P\Delta : P \to C \otimes P$. We define the left coaction-invariant subalgebra ${}^{\mathrm{co}C}P$ as in (1.18), and derive the left-sided version of (1.21). Hence, for any e-coaugmented coalgebra-Galois C-extension with *invertible canonical entwining*, the right coaction-invariant subalgebra coincides with the left coaction-invariant subalgebra:

$$P^{\mathrm{co}C} = \{p \in P \mid \Delta_P(p) = p \otimes e\} = \{p \in P \mid {}_P\Delta(p) = e \otimes p\} = {}^{\mathrm{co}C}P. \tag{1.25}$$

Finally, we need to assume one more condition on C-extensions to obtain a suitable definition: *equivariant projectivity*. It is a pivotal property that guarantees the projectivity of associated modules, and thus leads to index pairings between K-theory and K-homology. Putting together the aforementioned four conditions, we say that a coalgebra C-extension $B \subseteq P$ is *principal* [9] if:

(i) The canonical map $\mathsf{can} : P \otimes_B P \to P \otimes C$, $p \otimes_B p' \mapsto p\Delta_P(p')$, is bijective (Galois condition).

(ii) The right coaction is e-coaugmented for some group-like $e \in C$, i.e., $\Delta_P(1) = 1 \otimes e$.

(iii) The canonical entwining $\psi : C \otimes P \to P \otimes C$, $c \otimes p \mapsto \mathsf{can}(\mathsf{can}^{-1}(1 \otimes c)p)$, is bijective.

(iv) The algebra P is C-equivariantly projective as a left B-module, i.e., there exists a left B-linear and right C-colinear splitting of the multiplication map $B \otimes P \to P$.

In the framework of coalgebra extensions, the role of connections on principal bundles is played by strong connections [9]. Let P be an algebra and both a left and right e-coaugmented

C-comodule. (Note that the left and right coactions need not commute.) A *strong connection* is a linear map $\ell : C \to P \otimes P$ satisfying

$$\widetilde{\mathsf{can}} \circ \ell = 1 \otimes \mathrm{id}, \quad (\mathrm{id} \otimes \Delta_P) \circ \ell = (\ell \otimes \mathrm{id}) \circ \Delta, \quad ({}_P\Delta \otimes \mathrm{id}) \circ \ell = (\mathrm{id} \otimes \ell) \circ \Delta, \quad \ell(e) = 1 \otimes 1. \tag{1.26}$$

Here $\widetilde{\mathsf{can}} : P \otimes P \to P \otimes C$ is the lifting of can to $P \otimes P$. Assuming that there exists an invertible entwining $\psi : C \otimes P \to P \otimes C$ making P an entwined module, the first three equations of (1.26) read in the Heyneman-Sweedler type notation $c \mapsto \ell(c)^{\langle 1 \rangle} \otimes \ell(c)^{\langle 2 \rangle}$ as follows:

$$\ell(c)^{\langle 1 \rangle} \psi(e \otimes \ell(c)^{\langle 2 \rangle}) = \ell(c)^{\langle 1 \rangle} \ell(c)^{\langle 2 \rangle}{}_{(0)} \otimes \ell(c)^{\langle 2 \rangle}{}_{(1)} = 1 \otimes c, \tag{1.27}$$

$$\ell(c)^{\langle 1 \rangle} \otimes \psi(e \otimes \ell(c)^{\langle 2 \rangle}) = \ell(c)^{\langle 1 \rangle} \otimes \ell(c)^{\langle 2 \rangle}{}_{(0)} \otimes \ell(c)^{\langle 2 \rangle}{}_{(1)} = \ell(c_{(1)})^{\langle 1 \rangle} \otimes \ell(c_{(1)})^{\langle 2 \rangle} \otimes c_{(2)}, \tag{1.28}$$

$$\psi^{-1}(\ell(c)^{\langle 1 \rangle} \otimes e) \otimes \ell(c)^{\langle 2 \rangle} = \ell(c)^{\langle 1 \rangle}{}_{(-1)} \otimes \ell(c)^{\langle 1 \rangle}{}_{(0)} \otimes \ell(c)^{\langle 2 \rangle} = c_{(1)} \otimes \ell(c_{(2)})^{\langle 1 \rangle} \otimes \ell(c_{(2)})^{\langle 2 \rangle}. \tag{1.29}$$

Applying $\mathrm{id} \otimes \varepsilon$ to (1.27) yields the useful formula

$$\ell(c)^{\langle 1 \rangle} \ell(c)^{\langle 2 \rangle} = \varepsilon(c). \tag{1.30}$$

It is worthwhile to observe the left-right symmetry of principal extensions. We already noted (see (1.25)) the equality of the left and right coaction-invariant subalgebras. Now let us define the left canonical map as

$$\mathsf{can}_L : P \underset{B}{\otimes} P \ni p \otimes q \longmapsto p_{(-1)} \otimes p_{(0)} q \in C \otimes P. \tag{1.31}$$

One can check that it is related to the right canonical map can by the formula [10]

$$\psi \circ \mathsf{can}_L = \mathsf{can} \tag{1.32}$$

Also, if ℓ is a strong connection and $\widetilde{\mathsf{can}}_L := (\mathrm{id} \otimes m) \circ ({}_P\Delta \otimes \mathrm{id})$ is the lifted left canonical map, then $\widetilde{\mathsf{can}}_L \circ \ell = \mathrm{id} \otimes 1$. Hence

$$c \otimes p \longmapsto \ell(c)^{\langle 1 \rangle} \otimes \ell(c)^{\langle 2 \rangle} p \tag{1.33}$$

is a splitting of $\widetilde{\mathsf{can}}_L$ just as

$$p \otimes c \longmapsto p\ell(c)^{\langle 1 \rangle} \otimes \ell(c)^{\langle 2 \rangle} \tag{1.34}$$

is a splitting of $\widetilde{\mathsf{can}}$.

Lemma 1.2. *Let P be an object in the category ${}^C_e\mathbf{Alg}^C_e$ of all unital algebras with e-coaugmented left and right C-coactions. Assume that there exists an invertible entwining $\psi : C \otimes P \to P \otimes C$ making P an entwined module. Then, if P admits a strong connection ℓ, it is principal.*

Proof. Following [9], first we argue that

$$\sigma : P \ni p \longmapsto p_{(0)} \ell(p_{(1)})^{\langle 1 \rangle} \otimes \ell(p_{(1)})^{\langle 2 \rangle} \in B \otimes P \tag{1.35}$$

is a left B-linear splitting of the multiplication map. Indeed, $m \circ \sigma = \mathrm{id}$ because of (1.30), and the calculation

$$\psi(e \otimes p_{(0)} \ell(p_{(1)})^{\langle 1 \rangle}) \otimes \ell(p_{(1)})^{\langle 2 \rangle} = p_{(0)} \ell(p_{(1)})^{\langle 1 \rangle} \otimes e \otimes \ell(p_{(1)})^{\langle 2 \rangle} \tag{1.36}$$

obtained using (1.15) proves that $\sigma(P) \subseteq B \otimes P$. This splitting is evidently right C-colinear, so that its existence proves the equivariant projectivity.

Next, let us check that the formula

$$\mathsf{can}^{-1} : P \otimes C \longrightarrow P \underset{B}{\otimes} P, \quad p \otimes c \longmapsto p\ell(c)^{\langle 1\rangle} \underset{B}{\otimes} \ell(c)^{\langle 2\rangle}, \tag{1.37}$$

defines the inverse of the canonical map can, so that the coaction of C is Galois. It follows from (1.27) that

$$\mathsf{can}(\mathsf{can}^{-1}(p \otimes c)) = p\ell(c)^{\langle 1\rangle}\ell(c)^{\langle 2\rangle}{}_{(0)} \otimes \ell(c)^{\langle 2\rangle}{}_{(1)} = p \otimes c \tag{1.38}$$

On the other hand, taking advantage of (1.30) and (1.35), we see that

$$\mathsf{can}^{-1}(\mathsf{can}(p \underset{B}{\otimes} q)) = pq_{(0)}\ell(q_{(1)})^{\langle 1\rangle} \underset{B}{\otimes} \ell(q_{(1)})^{\langle 2\rangle} = p \underset{B}{\otimes} q_{(0)}\ell(q_{(1)})^{\langle 1\rangle}\ell(q_{(1)})^{\langle 2\rangle} = p \underset{B}{\otimes} q. \tag{1.39}$$

Thus the conditions (i) and (iv) of the principality of a C-extension are satisfied. Finally, Condition (ii) is simply assumed, and Condition (iii) follows from the uniqueness of an entwining that makes P an entwined module. $\square$

Note that, if there exists a strong connection ℓ, then (1.37) yields

$$\tau(c) = \ell(c)^{\langle 1\rangle} \underset{B}{\otimes} \ell(c)^{\langle 2\rangle}. \tag{1.40}$$

In the Heyneman-Sweedler type notation, we write $\tau(c) = \tau(c)^{[1]} \otimes_B \tau(c)^{[2]}$. Then the canonical entwining reads

$$\psi(c \otimes p) = \tau(c)^{[1]}(\tau(c)^{[2]}p)_{(0)} \otimes (\tau(c)^{[2]}p)_{(1)} = \ell(c)^{\langle 1\rangle}(\ell(c)^{\langle 2\rangle}p)_{(0)} \otimes (\ell(c)^{\langle 2\rangle}p)_{(1)}. \tag{1.41}$$

Remark 1.3. In [9], there is the converse statement: if P is principal, it admits a strong connection. Thus principal extensions can be characterized as these that admit a strong connection.

Recall now that classical principal bundles can be viewed as functors transforming finite-dimentional vector spaces into associated vector bundles. Analogously, one can prove that a principal C-extension $B \subseteq P$ defines a functor from the category of finite-dimensional left C-comodules into the category of finitely generated projective left B-modules [9]. Explicitly, if V is a left C-comodule with coaction ${}_V\Delta$, this functor assigns to it the cotensor product

$$P \underset{C}{\Box} V := \{\textstyle\sum_i p_i \otimes v_i \in P \otimes V \mid \sum_i \Delta_P(p_i) \otimes v_i = \sum_i p_i \otimes {}_V\Delta(v_i)\}. \tag{1.42}$$

In particular, if $g \in C$ is a group-like element, ${}_{\mathbb{C}}\Delta(1) = g \otimes 1$ defines a 1-dimensional corepresentation, and $P\Box_C\mathbb{C} = \{p \in P \mid \Delta_P(p) = p \otimes g\}$ can be viewed as a noncommutative associated complex line bundle.

A fundamental special case of principal extensions is provided by *principal comodule algebras*. One assumes then that $C = H$ is a Hopf algebra with a bijective antipode S, the canonical map is bijective, and P is an H-equivariantly projective left B-module. This brings us in touch with compact quantum groups. Assume that A is the C^*-algebra of a compact quantum group in the sense of Woronowicz [29, 31], and H is its dense Hopf *-subalgebra spanned by the matrix coefficients of the irreducible unitary corepresentations. Let P be a unital C^*-algebra and

$\delta : P \to P \otimes_{\min} A$ an injective C^*-algebraic right coaction of A on P. (See [1, Definition 0.2] for a general definition and [5, Definition 1] for the special case of compact quantum groups.) Here $\otimes_{\min}$ denotes the minimal C^*-completion of the algebraic tensor product $P \otimes A$. One can easily check that the subalgebra $PW_\delta(P) \subseteq P$ of elements for which the coaction lands in $P \otimes H$, i.e.,

$$PW_\delta(P) := \{p \in P \mid \delta(p) \in P \otimes H\}, \tag{1.43}$$

is an H-comodule algebra. It follows from results of [5] and [24] that $PW_\delta(P)$ is dense in P. It is straightforward to verify that the operation $P \mapsto PW_\delta(P)$ is a functor commuting with taking fibre products (pullbacks) [3]. Note also that $P^{\mathrm{co}A}$ is a C^*-algebra and $P^{\mathrm{co}A} = PW_\delta(P)^{\mathrm{co}H}$. We call $PW_\delta(P)$ the *Peter-Weyl comodule algebra* associated to the C^*-coaction δ.

1.4 The standard Hopf fibration of quantum SU(2)

The standard quantum Hopf fibration is given by an action of U(1) on the quantum group $\mathrm{SU}_q(2)$, $q \in (0,1)$. The coordinate ring of $\mathcal{O}(\mathrm{SU}_q(2))$ is generated by α, β, γ, δ with relations

$$\alpha\beta = q\beta\alpha, \quad \alpha\gamma = q\gamma\alpha, \quad \beta\delta = q\delta\beta, \quad \gamma\delta = q\delta\gamma, \quad \beta\gamma = \gamma\beta, \tag{1.44}$$
$$\alpha\delta - q\beta\gamma = 1, \quad \delta\alpha - q^{-1}\beta\gamma = 1, \tag{1.45}$$

and involution $\alpha^* = \delta$, $\beta^* = -q\gamma$. It is a Hopf *-algebra with comultiplication Δ, counit ε, and antipode S given by

$$\Delta(\alpha) = \alpha \otimes \alpha + \beta \otimes \gamma, \quad \Delta(\beta) = \alpha \otimes \beta + \beta \otimes \delta, \tag{1.46}$$
$$\Delta(\gamma) = \gamma \otimes \alpha + \delta \otimes \gamma, \quad \Delta(\delta) = \gamma \otimes \beta + \delta \otimes \delta, \tag{1.47}$$
$$\varepsilon(\alpha) = \varepsilon(\delta) = 1, \quad \varepsilon(\beta) = \varepsilon(\gamma) = 0, \tag{1.48}$$
$$S(\alpha) = \delta, \quad S(\beta) = -q^{-1}\beta, \quad S(\gamma) = -q\gamma, \quad S(\delta) = \alpha. \tag{1.49}$$

Let $\mathcal{O}(\mathrm{U}(1))$ denote the commutative and cocommutative Hopf *-algebra generated by the unitary grouplike element v. There is a Hopf *-algebra surjection $\pi : \mathcal{O}(\mathrm{SU}_q(2)) \to \mathcal{O}(\mathrm{U}(1))$ given by $\pi(\alpha) = v$, $\pi(\delta) = v^{-1}$ and $\pi(\beta) = \pi(\gamma) = 0$. Setting $\Delta_R := (\mathrm{id} \otimes \pi) \circ \Delta$, we obtain a right $\mathcal{O}(\mathrm{U}(1))$-coaction on $\mathcal{O}(\mathrm{SU}_q(2))$. On generators, the coaction reads

$$\Delta_R(\alpha) = \alpha \otimes v, \quad \Delta_R(\beta) = \beta \otimes v^{-1}, \quad \Delta_R(\gamma) = \gamma \otimes v, \quad \Delta_R(\delta) = \delta \otimes v^{-1}. \tag{1.50}$$

The *-subalgebra of coaction invariants defines the coordinate ring of the standard Podleś sphere [23]:

$$\mathcal{O}(\mathrm{S}^2_q) := \mathcal{O}(\mathrm{SU}_q(2))^{\mathrm{co}\mathcal{O}(\mathrm{U}(1))} = \{a \in \mathcal{O}(\mathrm{SU}_q(2)) \mid \Delta_R(a) = a \otimes 1\}. \tag{1.51}$$

One can prove that $\mathcal{O}(\mathrm{S}^2_q)$ is isomorphic to the *-algebra generated by B and the hermitean element A satifying the relations

$$AB = q^2 BA, \quad B^*B = A - A^2, \quad BB^* = q^2 A - q^4 A^2. \tag{1.52}$$

An isomorphism is explicitly given by the formulas $A = -q^{-1}\beta\gamma$ and $B = -\beta\alpha$. The irreducible Hilbert space representations of $\mathcal{O}(\mathrm{S}^2_q)$ are given by

$$\rho_0(A) = \rho_0(B) = 0, \quad \rho_0(1) = 1 \quad \text{on } \mathcal{H} = \mathbb{C}, \tag{1.53}$$
$$\rho_+(A)e_n = q^{2n}e_n, \quad \rho_+(B)e_n = q^n(1 - q^{2n})^{1/2}e_{n-1} \quad \text{on } \mathcal{H} = \ell_2(\mathbb{N}). \tag{1.54}$$

Here $\{e_n \mid n = 0, 1, \ldots\}$ is an orthonormal basis of $\ell_2(\mathbb{N})$.

Recall that the universal C^*-algebra of a complex *-algebra is the C^*-completion with respect to the universal C^*-norm given by the supremum of the operator norms over all bounded *-representations (if the supremum exists). Let $\mathcal{C}(\mathrm{S}^2_q)$ denote the universal C^*-algebra generated by A and B. From the above representations, it follows that

$$\mathcal{C}(\mathrm{S}^2_q) \;\cong\; \mathcal{K}(\ell_2(\mathbb{N})) \oplus \mathbb{C} \;\subseteq\; \mathcal{B}(\ell_2(\mathbb{N})). \tag{1.55}$$

Here $\mathcal{K}(\ell_2(\mathbb{N}))$ and $\mathcal{B}(\ell_2(\mathbb{N}))$ denote the C^*-algebras of compact and bounded operators on the Hilbert space $\ell_2(\mathbb{N})$, respectively. The isomorphism (1.55) implies that $\mathrm{K}_0(\mathcal{C}(\mathrm{S}^2_q)) \cong \mathbb{Z} \oplus \mathbb{Z}$, where one generator of K-theory is given by the class of the unit $1 \in \mathcal{C}(\mathrm{S}^2_q)$, and the other by the class of the 1-dimensional projection onto $\mathbb{C}e_0 \subseteq \ell_2(\mathbb{N})$.

Furthermore, $\mathrm{K}^0(\mathcal{C}(\mathrm{S}^2_q)) \;\cong\; \mathbb{Z} \oplus \mathbb{Z}$. We identify one generator of K-homology with the class of the pair of representations $[(\mathrm{id}, \varepsilon)]$, where $\mathrm{id}(k+\alpha) = k+\alpha$ and $\varepsilon(k+\alpha) = \alpha$ for all $k+\alpha \in \mathcal{K}(\ell_2(\mathbb{N}))\oplus\mathbb{C}$. The other generator can be given by the class of the pair of representations $[(\varepsilon, \varepsilon_0)]$ with the (non-unital) representation ε_0 of $\mathcal{K}(\ell_2(\mathbb{N})) \oplus \mathbb{C}$ defined by $\varepsilon_0(k+\alpha) = \alpha s s^*$, where

$$s : \ell_2(\mathbb{N}) \longrightarrow \ell_2(\mathbb{N}), \quad s e_n = e_{n+1}, \tag{1.56}$$

denotes the unilateral shift on $\ell_2(\mathbb{N})$. (See [20] for a detailed treatment of the K-homology and K-theory of Podleś spheres.)

We shall also consider the coordinate ring of the quantum disc $\mathcal{O}(\mathrm{D}_q)$ generated by z and z^* with relation

$$z^*z - q^2 zz^* = 1 - q^2. \tag{1.57}$$

Its bounded irreducible Hilbert space representations are given by

$$\mu_\theta(z) = \mathrm{e}^{\mathrm{i}\theta} \quad \text{on } \mathcal{H} = \mathbb{C}, \quad \theta \in [0, 2\pi), \tag{1.58}$$

$$\mu(z)e_n = (1 - q^{2(n+1)})^{1/2} e_{n+1} \quad \text{on } \mathcal{H} = \ell_2(\mathbb{N}). \tag{1.59}$$

It has been shown in [17] that the universal C^*-algebra of $\mathcal{O}(\mathrm{D}_q)$ is isomorphic to the Toeplitz algebra given as the universal C^*-algebra generated by the unilateral shift s of Equation (1.56). The representation μ defines then an embedding of $\mathcal{O}(\mathrm{D}_q)$ into $\mathcal{T}$.

Let $\mathcal{C}(\mathrm{S}^1)$ denote the C^*-algebra of continuous functions on the circle S^1, and let $u = \mathrm{e}^{\mathrm{i}t}$ be its generator. The Toeplitz algebra gives rise to the following short exact sequence of C^*-algebras:

$$0 \longrightarrow \mathcal{K}(\ell_2(\mathbb{N})) \longrightarrow \mathcal{T} \xrightarrow{\sigma} \mathcal{C}(\mathrm{S}^1) \longrightarrow 0. \tag{1.60}$$

Here the so-called symbol map $\sigma : \mathcal{T} \to \mathcal{C}(\mathrm{S}^1)$ is given by $\sigma(s) = u$. Since $s - \mu(z)$ belongs to $\mathcal{K}(\ell_2(\mathbb{N}))$, it follows in particular that $\sigma(\mu(z)) = u$. For later use, we state the following auxiliary lemma:

Lemma 1.4. *$\mathcal{K}(\ell_2(\mathbb{N})) \cap \mathcal{O}(\mathrm{D}_q)$ is dense in $\mathcal{K}(\ell_2(\mathbb{N}))$.*

Proof. Let $\mathcal{I}$ denote the ideal in $\mathcal{O}(\mathrm{D}_q)$ generated by the element $y := 1 - zz^* \in \mathcal{O}(\mathrm{D}_q)$. Note that $ye_n = q^{2n}e_n$. Consequently, $y \in \mathcal{K}(\ell_2(\mathbb{N}))$, so that $\mathcal{I} \subseteq \mathcal{K}(\ell_2(\mathbb{N})) \cap \mathcal{O}(\mathrm{D}_q)$. We shall prove that $\mathcal{I}$ is dense in $\mathcal{K}(\ell_2(\mathbb{N}))$.

Since the closure $\bar{\mathcal{I}}$ of $\mathcal{I}$ is a C^*-algebra, $f(y) \in \bar{\mathcal{I}}$ for all continuous functions f on $\mathrm{spec}(y) = \{q^{2n} \mid n \in \mathbb{N}\} \cup \{0\}$. In particular,

$$|z^m|^{-1} = (\prod_{k=1}^{m}(1-q^{2k}y))^{-1/2} \in \bar{\mathcal{I}}, \quad m \in \mathbb{N}, \ \ m > 0, \tag{1.61}$$

and $\chi_n(y) \in \bar{\mathcal{I}}$, where

$$\chi_n(t) := \left\{ \begin{array}{ll} 0 & \text{for } \ t \in \mathrm{spec}(y) \setminus \{q^{2n}\} \\ 1 & \text{for } \ t = q^{2n} \end{array} \right. , \qquad n \in \mathbb{N}. \tag{1.62}$$

Hence $E_{n+m,n} := z^m|z^m|^{-1}\chi_n(y) \in \bar{\mathcal{I}}$ for all $n, m \in \mathbb{N}$, and $E_{n-k,n} := \chi_{n-k}(y)|z^k|^{-1}z^{*k} \in \bar{\mathcal{I}}$ for $n, k \in \mathbb{N}$, $1 \le k \le n$. Then we can write $E_{n+m,n}e_j = \delta_{nj}e_{n+m}$ and $E_{n-k,n}e_j = \delta_{nj}e_{n-k}$, where δ_{ij} denotes the Kronecker delta. Since $\mathcal{K}(\ell_2(\mathbb{N}))$ is the C^*-algebra generated by the "elementary matrices" $E_{n+m,n}$ and $E_{n-k,n}$, we conclude that $\bar{\mathcal{I}} = \mathcal{K}(\ell_2(\mathbb{N}))$. $\square$

Now let us consider the associated quantum line bundles as finitely generated projective modules. They are defined by the 1-dimensional corepresentations $\mathbb{C} \ni 1 \mapsto v^{-N} \otimes 1$, $N \in \mathbb{Z}$, as follows:

$$M_N := \{p \in \mathcal{O}(\mathrm{SU}_q(2)) \mid \Delta_R(p) = p \otimes v^{-N}\}. \tag{1.63}$$

Since Δ_R is a morphism of algebras, M_N is an $\mathcal{O}(\mathrm{S}_q^2)$-bimodule. Our next step is to determine explicitly projections describing these projective modules.

For $l \in \frac{1}{2}\mathbb{N}$ and $i, j = -l, -l+1, \ldots, l$, let t_{ij}^l denote the matrix elements of the irreducible unitary corepresentations of $\mathcal{O}(\mathrm{SU}_q(2))$, that is,

$$\Delta(t_{ij}^l) = \sum_{k=-l}^{l} t_{ik}^l \otimes t_{kj}^l, \qquad \sum_{k=-l}^{l} t_{ki}^{l*} t_{kj}^l = \sum_{k=-l}^{l} t_{ik}^l t_{jk}^{l*} = \delta_{ij}. \tag{1.64}$$

By the Peter-Weyl theorem for compact quantum groups [30], $\mathcal{O}(\mathrm{SU}_q(2)) = \oplus_{l \in \frac{1}{2}\mathbb{N}} \oplus_{i,j=-l}^{l} \mathbb{C} t_{ij}^l$. From the explicit description of t_{ij}^l [18, Section 4.2.4] and the definition of Δ_R, it follows that $\Delta_R(t_{ij}^l) = t_{ij}^l \otimes v^{-2j}$, so that $t_{ij}^l \in M_{2j}$. It can be shown [15, 25] that $t_{ij}^{|j|}$, $i = -|j|, \ldots, |j|$ generate M_{2j} as a left $\mathcal{O}(\mathrm{S}_q^2)$-module and $M_{2j} \cong \mathcal{O}(\mathrm{S}_q^2)^{2|j|+1} E_{2j}$ for all $j \in \frac{1}{2}\mathbb{Z}$, where

$$E_{2j} = \begin{pmatrix} t_{-|j|,j}^{|j|} \\ \vdots \\ t_{|j|,j}^{|j|} \end{pmatrix} \begin{pmatrix} t_{-|j|,j}^{|j|*}, & \cdots, & t_{|j|,j}^{|j|*} \end{pmatrix} \in \mathrm{Mat}_{2|j|+1}(\mathcal{O}(\mathrm{S}_q^2)). \tag{1.65}$$

Finally, note that $t_{i,j}^{|j|} t_{k,j}^{|j|*}$ is indeed in $\mathcal{O}(\mathrm{S}_q^2)$ because

$$\Delta_R(t_{i,j}^{|j|} t_{k,j}^{|j|*}) = t_{i,j}^{|j|} t_{k,j}^{|j|*} \otimes v^{-2j} v^{-2j*} = t_{i,j}^{|j|} t_{k,j}^{|j|*} \otimes 1. \tag{1.66}$$

Also, it is clear that $E_{2j}^* = E_{2j}$ and $E_{2j}^2 = E_{2j}$, so that E_{2j} is a projection.

2 The principality of one-surjective pullbacks

2.1 The pullbacks of algebras with left and right coactions

The purpose of this section is to define an ambient category for pullback diagrams appearing in the next section. Let P be a unital algebra equipped with both a right coaction $\Delta_P : P \to P \otimes C$ and a left coaction ${}_P\Delta : P \to C \otimes P$ of the same coalgebra C. We do *not* assume that these coactions commute, but we do assume that they are coaugmented by the same group-like element $e \in C$, i.e., $\Delta_P(1) = 1 \otimes e$ and ${}_P\Delta(1) = e \otimes 1$. For a fixed coalgebra C and a group-like $e \in C$, we consider the category ${}^C_e\mathbf{Alg}^C_e$ of all such unital algebras with e-coaugmented left and right C-coactions. Here morphisms are bicolinear algebra homomorphisms.

Since we work over a field, this category is evidently closed under any pullbacks. If $\pi_1 : P_1 \to P_{12}$ and $\pi_2 : P_2 \to P_{12}$ are morphisms in ${}^C_e\mathbf{Alg}^C_e$, then the fibre product algebra $P := P_1 \times_{(\pi_1,\pi_2)} P_2$ becomes a right C-comodule via

$$\Delta_P(p, q) = (p_{(0)}, 0) \otimes p_{(1)} + (0, q_{(0)}) \otimes q_{(1)}, \tag{2.1}$$

and a left C-comodule via

$${}_P\Delta(p, q) = p_{(-1)} \otimes (p_{(0)}, 0) + q_{(-1)} \otimes (0, q_{(0)}). \tag{2.2}$$

Also, it is clear that $\Delta_P(1, 1) = (1, 1) \otimes e$ and ${}_P\Delta(1, 1) = e \otimes (1, 1)$.

In the following lemma, we prove that any surjective morphism in ${}^C_e\mathbf{Alg}^C_e$ whose domain is a principal extension can be split by a left colinear map and by a right colinear map (not necessarily by a bicolinear map). Note that the first part of the lemma is proved much the same way as in the Hopf-Galois case [14, Lemma 3.1]:

Lemma 2.1. *Let $\pi : P \to Q$ be a surjective morphism in the category ${}^C_e\mathbf{Alg}^C_e$ of unital algebras with e-coaugmented left and right C-coactions. If P is principal, then:*

(i) The induced map $\pi^{\mathrm{co}C} : P^{\mathrm{co}C} \to Q^{\mathrm{co}C}$ is surjective.

(ii) There exists a unital right C-colinear splitting of π.

(iii) There exists a unital left C-colinear splitting of π.

(iv) Q is principal.

Furthermore, if $Q' \in {}^C_e\mathbf{Alg}^C_e$, $Q' \subseteq Q$, is principal, then so is $\pi^{-1}(Q')$.

Proof. It follows from the right colinearity and surjectivity of π that $\pi(P^{\mathrm{co}C}) \subseteq Q^{\mathrm{co}C}$. To prove the converse inclusion, we take advantage of the left $P^{\mathrm{co}C}$-linear retraction of the inclusion $P^{\mathrm{co}C} \subseteq P$ that was used to prove [9, Theorem 2.5(3)]:

$$\sigma_\varphi : P \longrightarrow P^{\mathrm{co}C}, \quad \sigma_\varphi(p) := p_{(0)}\ell(p_{(1)})^{\langle 1\rangle}\varphi(\ell(p_{(1)})^{\langle 2\rangle})\,. \tag{2.3}$$

Here ℓ is a strong connection on P and φ is any unital linear functional on P. It follows from (1.35) that $\sigma_\varphi(p) \in P^{\mathrm{co}C}$. If $\pi(p) \in Q^{\mathrm{co}C}$, then $\sigma_\varphi(p)$ is a desired element of $P^{\mathrm{co}C}$ that is

mapped by π to $\pi(p)$. Indeed, since $\pi(p_{(0)}) \otimes p_{(1)} = \pi(p)_{(0)} \otimes \pi(p)_{(1)} = \pi(p) \otimes e$, using the unitality of π, φ, and $\ell(e) = 1 \otimes 1$, we compute

$$\pi(\sigma_\varphi(p)) = \pi(p_{(0)})\pi(\ell(p_{(1)})^{\langle 1\rangle})\varphi(\ell(p_{(1)})^{\langle 2\rangle}) = \pi(p). \tag{2.4}$$

To show the second assertion, let us choose any unital k-linear splitting of $\pi\lceil_{P^{\mathrm{co}C}}$ and denote it by $\alpha^{\mathrm{co}C}$. We want to prove that the formula

$$\alpha_R(q) := \alpha^{\mathrm{co}C}(q_{(0)}\pi(\ell(q_{(1)})^{\langle 1\rangle}))\ell(q_{(1)})^{\langle 2\rangle} \tag{2.5}$$

defines a unital right colinear splitting of π. Since π is surjective, we can write $q = \pi(p)$. Then, using properties of π, we obtain:

$$\begin{aligned} q_{(0)}\,\pi(\ell(q_{(1)})^{\langle 1\rangle})\otimes \ell(q_{(1)})^{\langle 2\rangle} &= \pi(p)_{(0)}\,\pi(\ell(\pi(p)_{(1)})^{\langle 1\rangle})\otimes \ell(\pi(p)_{(1)})^{\langle 2\rangle} \\ &= \pi(p_{(0)})\pi(\ell(p_{(1)})^{\langle 1\rangle})\otimes \ell(p_{(1)})^{\langle 2\rangle} \\ &= \pi(p_{(0)}\ell(p_{(1)})^{\langle 1\rangle})\otimes \ell(p_{(1)})^{\langle 2\rangle}. \end{aligned} \tag{2.6}$$

Now it follows from (1.35) that the above tensor is in $Q^{\mathrm{co}C} \otimes P$. Hence α_R is well defined. It is straightforward to verify that α_R is unital, right colinear, and splits π. (Note that, since $q \in Q^{\mathrm{co}C}$ implies $q_{(0)} \otimes q_{(1)} = q \otimes e$, we have $\alpha^{\mathrm{co}C} = \alpha_R\lceil_{Q^{\mathrm{co}C}}$.) The third assertion can be proven in an analogous manner.

To prove (iv), we first show that the inverse of the canonical map $\mathsf{can}_Q : Q \otimes_{Q^{\mathrm{co}C}} Q \to Q \otimes C$ (see (1.19)) is given by

$$\mathsf{can}_Q^{-1} : Q \otimes C \longrightarrow Q \underset{Q^{\mathrm{co}C}}{\otimes} Q, \quad q \otimes c \longmapsto q\pi(\ell(c)^{\langle 1\rangle}) \underset{Q^{\mathrm{co}C}}{\otimes} \pi(\ell(c)^{\langle 2\rangle}). \tag{2.7}$$

Using the properties of π and ℓ, we get

$$\begin{aligned} (\mathsf{can}_Q \circ \mathsf{can}_Q^{-1})\,(\pi(p) \otimes c) &= \mathsf{can}_Q\Big(\pi(p\ell(c)^{\langle 1\rangle}) \underset{Q^{\mathrm{co}C}}{\otimes} \pi(\ell(c)^{\langle 2\rangle})\Big) \\ &= \pi\left(p\,\ell(c)^{\langle 1\rangle}\,\ell(c)^{\langle 2\rangle}{}_{(0)}\right) \otimes \ell(c)^{\langle 2\rangle}{}_{(1)} \\ &= \pi(p) \otimes c. \end{aligned} \tag{2.8}$$

Similarly,

$$\begin{aligned} (\mathsf{can}_Q^{-1} \circ \mathsf{can}_Q)\Big(\pi(p) \underset{Q^{\mathrm{co}C}}{\otimes} \pi(p')\Big) &= \mathsf{can}_Q^{-1}\Big(\pi(pp'_{(0)}) \otimes p'_{(1)}\Big) \\ &= \pi(pp'_{(0)}\ell(p'_{(1)})^{\langle 1\rangle}) \underset{Q^{\mathrm{co}C}}{\otimes} \pi(\ell(p'_{(1)})^{\langle 2\rangle}) \qquad (2.9) \\ &= \pi(p) \underset{Q^{\mathrm{co}C}}{\otimes} \pi(p'_{(0)}\ell(p'_{(1)})^{\langle 1\rangle}\ell(p'_{(1)})^{\langle 2\rangle}) \\ &= \pi(p) \underset{Q^{\mathrm{co}C}}{\otimes} \pi(p'). \qquad (2.10) \end{aligned}$$

Here we used the fact that $\pi(p'_{(0)}\ell(p'_{(1)})^{\langle 1\rangle})\otimes\ell(p'_{(1)})^{\langle 2\rangle} \in Q^{\mathrm{co}C}\otimes P$. Hence the extension $Q^{\mathrm{co}C} \subseteq Q$ is Galois, and we have the canonical entwining $\psi_Q : C \otimes Q \to Q \otimes C$.

Our next aim is to prove that ψ_Q is bijective. We know by assumption that the canonical entwining $\psi_P : C \otimes P \to P \otimes C$ is invertible. To determine its inverse, recall that the left

and right coactions are given by $\psi_P^{-1}(p\otimes e)$ and $\psi_P(e\otimes p)$, respectively. Then apply (1.15) to compute

$$\begin{aligned}\psi_P\Big((p\ell(c)^{\langle 1\rangle})_{(-1)}\otimes(p\ell(c)^{\langle 1\rangle})_{(0)}\,\ell(c)^{\langle 2\rangle}\Big) &= p\ell(c)^{\langle 1\rangle}\psi_P\Big(e\otimes\ell(c)^{\langle 2\rangle}\Big)\\ &= p\ell(c)^{\langle 1\rangle}\ell(c)^{\langle 2\rangle}{}_{(0)}\otimes\ell(c)^{\langle 2\rangle}{}_{(1)}\\ &= \ p\otimes c. \end{aligned}\tag{2.11}$$

Hence $\psi_P^{-1}(p\otimes c)=(p\ell(c)^{\langle 1\rangle})_{(-1)}\otimes(p\ell(c)^{\langle 1\rangle})_{(0)}\,\ell(c)^{\langle 2\rangle}$. On the other hand,

$$\begin{aligned}\psi_Q(c\otimes\pi(p)) &= \pi(\ell(c)^{\langle 1\rangle})\big(\pi(\ell(c)^{\langle 2\rangle})\pi(p)\big)_{(0)}\otimes\big(\pi(\ell(c)^{\langle 2\rangle})\pi(p)\big)_{(1)}\\ &= \pi\big(\ell(c)^{\langle 1\rangle}\big)\,\pi\big((\ell(c)^{\langle 2\rangle}p)_{(0)}\big)\otimes(\ell(c)^{\langle 2\rangle}p)_{(1)}\\ &= (\pi\otimes\mathrm{id})\big(\psi_P(c\otimes p)\big), \end{aligned}\tag{2.12}$$

$$\begin{aligned}(\mathrm{id}\otimes\pi)\big(\psi_P^{-1}(p\otimes c)\big) &= (p\,\ell(c)^{\langle 1\rangle})_{(-1)}\otimes\pi\big((p\,\ell(c)^{\langle 1\rangle})_{(0)}\big)\,\pi\big(\ell(c)^{\langle 2\rangle}\big)\\ &= \big(\pi(p\,\ell(c)^{\langle 1\rangle})\big)_{(-1)}\otimes\big(\pi(p\,\ell(c)^{\langle 1\rangle})\big)_{(0)}\,\pi(\ell(c)^{\langle 2\rangle})\\ &= {}_Q\Delta\big(\pi(p)\pi(\ell(c)^{\langle 1\rangle})\big)\pi(\ell(c)^{\langle 2\rangle}). \end{aligned}\tag{2.13}$$

The second part of the above computation implies that the assignment

$$\psi_Q^{-1}:Q\otimes C\longrightarrow C\otimes Q,\quad \pi(p)\otimes c\longmapsto(\mathrm{id}\otimes\pi)(\psi_P^{-1}(p\otimes c))\tag{2.14}$$

is well defined. Now it follows from the first part that ψ_Q^{-1} is the inverse of ψ_Q:

$$\psi_Q\Big(\psi_Q^{-1}\big(\pi(p)\otimes c\big)\Big)=\psi_Q\Big((\mathrm{id}\otimes\pi)\big(\psi_P^{-1}(p\otimes c)\big)\Big)=(\pi\otimes\mathrm{id})\Big(\psi_P\big(\psi_P^{-1}(p\otimes c)\big)\Big)=\pi(p)\otimes c,\tag{2.15}$$

$$\psi_Q^{-1}\Big(\psi_Q\big(c\otimes\pi(p)\big)\Big)=\psi_Q^{-1}\Big((\pi\otimes\mathrm{id})\big(\psi_P(c\otimes p)\big)\Big)=(\mathrm{id}\otimes\pi)\Big(\psi_P^{-1}\big(\psi_P(c\otimes p)\big)\Big)=c\otimes\pi(p).\tag{2.16}$$

On the other hand, we observe that $(\pi\otimes\pi)\circ\ell$ is a strong connection on Q. Combined with the just proven existence of a bijective entwining that makes Q an entwined module, it allows us to apply Lemma 1.2 and conclude the proof of *(iv)*.

To prove the final statement of the lemma, note first that $\pi^{-1}(Q')\in{}^C_e\mathbf{Alg}^C_e$. Next, observe that, if $\ell':C\to Q'\otimes Q'$ is a strong connection on Q', then it is also a strong connection on Q. Now, it follows from (1.41) that for any $q\in Q'$

$$\psi_Q(c\otimes q)=\ell'(c)^{\langle 1\rangle}\,\big(\ell'(c)^{\langle 2\rangle}q\big)_{(0)}\otimes\big(\ell'(c)^{\langle 2\rangle}q\big)_{(1)}\in Q'\otimes C.\tag{2.17}$$

Much the same way, it follows from the Q-analog of the formula following (2.11) that $\psi_Q^{-1}(Q'\otimes C)\subseteq C\otimes Q'$. Hence to see that ψ_P and ψ_P^{-1} restrict to $\pi^{-1}(Q')$, we can apply (2.12) and (2.14), respectively.

A key step now is to construct a strong connection on $\pi^{-1}(Q')$. Let α_R and α_L be, respectively, right and left colinear unital splittings of π. Their existence is guaranteed by the already proven *(ii)* and *(iii)*. The map $(\alpha_L\otimes\alpha_R)\circ\ell':C\to\pi^{-1}(Q)\otimes\pi^{-1}(Q)$ is bicolinear and satisfies

$$\alpha_L(\ell'(e)^{\langle 1\rangle})\otimes\alpha_R(\ell'(e)^{\langle 2\rangle})=1\otimes 1.\tag{2.18}$$

However,

$$1\otimes c-\big(\widetilde{\mathsf{can}}\circ(\alpha_L\otimes\alpha_R)\circ\ell'\big)(c)=1\otimes c-\alpha_L(\ell'(c_{(1)})^{\langle 1\rangle})\alpha_R(\ell'(c_{(1)})^{\langle 2\rangle})\otimes c_{(2)}\neq 0. \tag{2.19}$$

To solve this problem, we apply to it the splitting of the lifted canonical map given by a strong connection ℓ (see (1.34)), and add to $(\alpha_L\otimes\alpha_R)\circ\ell'$:

$$\ell_R(c):=(\alpha_L\otimes\alpha_R)(\ell'(c))+\ell(c)-\alpha_L(\ell'(c_{(1)})^{\langle 1\rangle})\alpha_R(\ell'(c_{(1)})^{\langle 2\rangle})\ell(c_{(2)}^{\langle 1\rangle})\otimes\ell(c_{(2)}^{\langle 2\rangle}). \tag{2.20}$$

Now $\widetilde{\mathsf{can}}\circ\ell_R=1\otimes\mathrm{id}$, as needed. Also, $\ell_R(e)=1\otimes 1$ and $((\pi\otimes\mathrm{id})\circ\ell_R)(C)\subseteq Q'\otimes P$. The right colinearity of ℓ_R is clear. To check the left colinearity of ℓ_R, using the fact that P is a ψ_P entwined and e-coaugmented module, we show that $(m_P\circ(\alpha_L\otimes\alpha_R)\circ\ell')*\ell$ is left colinear. (Here m_P is the multiplication of P.) First we note that

$$({}_P\Delta\otimes\mathrm{id})\circ\big((m_P\circ(\alpha_L\otimes\alpha_R)\circ\ell')*\ell\big)=\big(\mathrm{id}\otimes(m_P\circ(\alpha_L\otimes\alpha_R)\circ\ell')*\ell\big)\circ\Delta \tag{2.21}$$

is equivalent to

$$\begin{aligned}&\alpha_L(\ell'(c_{(1)})^{\langle 1\rangle})\alpha_R(\ell'(c_{(1)})^{\langle 2\rangle})\ell(c_{(2)})^{\langle 1\rangle}\otimes e\otimes\ell(c_{(2)})^{\langle 2\rangle}\\&\qquad=\psi_P\Big(c_{(1)}\otimes\alpha_L(\ell'(c_{(2)})^{\langle 1\rangle})\alpha_R(\ell'(c_{(2)})^{\langle 2\rangle})\ell(c_{(3)})^{\langle 1\rangle}\Big)\otimes\ell(c_{(3)})^{\langle 2\rangle}.\end{aligned} \tag{2.22}$$

Since $c_{(1)}\otimes\alpha_L(\ell'(c_{(2)})^{\langle 1\rangle})\otimes\ell'(c_{(2)})^{\langle 2\rangle}=\psi_P^{-1}\big(\alpha_L(\ell'(c)^{\langle 1\rangle})\otimes e\big)\otimes\ell'(c)^{\langle 2\rangle}$, we obtain

$$\begin{aligned}&\psi_P\Big(c_{(1)}\otimes\alpha_L(\ell'(c_{(2)})^{\langle 1\rangle})\alpha_R(\ell'(c_{(2)})^{\langle 2\rangle})\ell(c_{(3)})^{\langle 1\rangle}\Big)\otimes\ell(c_{(3)})^{\langle 2\rangle}\\&\qquad=\alpha_L(\ell'(c_{(1)})^{\langle 1\rangle})\,\psi_P\Big(e\otimes\alpha_R(\ell'(c_{(1)})^{\langle 2\rangle})\ell(c_{(2)})^{\langle 1\rangle}\Big)\otimes\ell(c_{(2)})^{\langle 2\rangle}\\&\qquad=\alpha_L(\ell'(c_{(1)})^{\langle 1\rangle})\alpha_R(\ell'(c_{(1)})^{\langle 2\rangle})\,\psi_P\Big(c_{(2)}\otimes\ell(c_{(3)})^{\langle 1\rangle}\Big)\otimes\ell(c_{(3)})^{\langle 2\rangle}\\&\qquad=\alpha_L(\ell'(c_{(1)})^{\langle 1\rangle})\alpha_R(\ell'(c_{(1)})^{\langle 2\rangle})\,\psi_P\Big(\psi_P^{-1}\big(\ell(c_{(2)})^{\langle 1\rangle}\otimes e\big)\Big)\otimes\ell(c_{(2)})^{\langle 2\rangle}\\&\qquad=\alpha_L(\ell'(c_{(1)})^{\langle 1\rangle})\alpha_R(\ell'(c_{(1)})^{\langle 2\rangle})\,\ell(c_{(2)})^{\langle 1\rangle}\otimes e\otimes\ell(c_{(2)})^{\langle 2\rangle}.\end{aligned} \tag{2.23}$$

Hence ℓ_R is a strong connection with the property $\ell_R(C)\subseteq\pi^{-1}(Q')\otimes P$. In a similar manner, we construct a strong connection ℓ_L with the property $\ell_L(C)\subset P\otimes\pi^{-1}(Q')$. Now we need to apply the splitting of the left lifted canonical map given by ℓ (see (1.33)) to derive the formula

$$\ell_L:=(\alpha_L\otimes\alpha_R)\circ\ell'+\ell-\ell*(m_P\circ(\alpha_L\otimes\alpha_R)\circ\ell'). \tag{2.24}$$

It is clear that $\ell_L(e)=1\otimes 1$ and $\ell_L(C)\subseteq P\otimes\pi^{-1}(Q')$. A computation similar to (2.23) shows the right colinearity of ℓ_L. Since furthermore $\psi_P(1\otimes c)=c\otimes 1$ for any $c\in C$ and $\widetilde{\mathsf{can}}=\psi_P\circ\widetilde{\mathsf{can}}_L$, we obtain

$$\widetilde{\mathsf{can}}(\ell_L(c))=\psi_P\big(\widetilde{\mathsf{can}}_L(\ell(c))\big)=\psi_P(c\otimes 1)=1\otimes c. \tag{2.25}$$

Hence ℓ_L is a desired strong connection. Plugging it into (2.20) instead of ℓ, we get a strong connection

$$\ell_{LR}=(\alpha_L\otimes\alpha_R)\circ\ell'+\ell_L-(m_p\circ(\alpha_L\otimes\alpha_R)\circ\ell')*\ell_L \tag{2.26}$$

with the property $\ell_{LR}\subseteq\pi^{-1}(Q')\otimes\pi^{-1}(Q')$. Applying now Lemma 1.2 ends the proof of this lemma. □

2.2 The one-surjective pullbacks of principal coactions are principal

Our goal now is to show that the subcategory of principal extensions is closed under one-surjective pullbacks. Here the right coaction is the coaction defining a principal extension and the left coaction is the one defined by the inverse of the canonical entwining (see (1.24)). With this structure, principal extensions form a full subcategory of ${}^{C}_{e}\mathbf{Alg}^{C}_{e}$. The following theorem is the main result of this paper generalizing the theorem of [14] on the pullback of surjections of principal comodule algebras:

Theorem 2.2. *Let C be a coalgebra, $e \in C$ a group-like element, and P the pullback of $\pi_1 : P_1 \to P_{12}$ and $\pi_2 : P_2 \to P_{12}$ in the category ${}^{C}_{e}\mathbf{Alg}^{C}_{e}$ of unital algebras with e-coaugmented left and right C-coactions. If π_1 or π_2 is surjective and both P_1 and P_2 are principal e-coaugmented C-extensions, then also P is a principal e-coaugmented C-extension.*

Proof. Without loss of generality, we assume that π_1 is surjective. We first show that P inherits an entwined structure from P_1 and P_2.

Lemma 2.3. *Let ψ_1 and ψ_2 denote the entwining structures of P_1 and P_2, respectively. Then P is an entwined module with an invertible entwining structure*

$$\psi = \psi_1 \circ (\mathrm{id} \otimes \mathrm{pr}_1) + \psi_2 \circ (\mathrm{id} \otimes \mathrm{pr}_2). \tag{2.27}$$

Here pr_1 *and* pr_2 *are morphisms of the pullback diagram as in* (1.3).

Proof. Our strategy is to construct a bijective map $\tilde{\psi} : C \otimes (P_1 \times P_2) \to (P_1 \times P_2) \otimes C$, and to show that it restricts to a bijective entwining on $C \otimes P$. We put

$$\tilde{\psi} := \psi_1 \circ (\mathrm{id} \otimes \tilde{\mathrm{pr}}_1) + \psi_2 \circ (\mathrm{id} \otimes \tilde{\mathrm{pr}}_2). \tag{2.28}$$

The symbols $\tilde{\mathrm{pr}}_1$ and $\tilde{\mathrm{pr}}_2$ stand for respective componentwise projections. Their restrictions to P yield pr_1 and pr_2. It is easy to check that the inverse of $\tilde{\psi}$ is given by

$$\tilde{\psi}^{-1} = \psi_1^{-1} \circ (\tilde{\mathrm{pr}}_1 \otimes \mathrm{id}) + \psi_2^{-1} \circ (\tilde{\mathrm{pr}}_2 \otimes \mathrm{id}) \tag{2.29}$$

To show that $\tilde{\psi}(C \otimes P) \subseteq P \otimes C$ and $\tilde{\psi}^{-1}(P \otimes C) \subseteq C \otimes P$, we note first that P_{12} and $\pi_2(P_2)$ are principal by Lemma 2.1(iv). Consequently, their canonical entwinings ψ_{12} and $\psi_{\pi_2(P_2)}$ are bijective. Furthermore, arguing as in the proof of Lemma 2.1, we see that $\psi_{\pi_2(P_2)} = \psi_{12}\restriction_{C\otimes\pi_2(P_2)}$ and $\psi^{-1}_{\pi_2(P_2)} = \psi_{12}^{-1}\restriction_{\pi_2(P_2)\otimes C}$. An advantage of having both summands in terms of ψ_{12} is that we can apply (2.12) to compute

$$\begin{aligned}\big((\pi_1 \circ \tilde{\mathrm{pr}}_1 - \pi_2 \circ \tilde{\mathrm{pr}}_2) \otimes \mathrm{id}\big) \circ \tilde{\psi} &= (\pi_1 \circ \tilde{\mathrm{pr}}_1 \otimes \mathrm{id}) \circ \psi_1 \circ (\mathrm{id} \otimes \tilde{\mathrm{pr}}_1) - (\pi_2 \circ \tilde{\mathrm{pr}}_2 \otimes \mathrm{id}) \circ \psi_1 \circ (\mathrm{id} \otimes \tilde{\mathrm{pr}}_1)\\ &\quad + (\pi_1 \circ \tilde{\mathrm{pr}}_1 \otimes \mathrm{id}) \circ \psi_2 \circ (\mathrm{id} \otimes \tilde{\mathrm{pr}}_2) - (\pi_2 \circ \tilde{\mathrm{pr}}_2 \otimes \mathrm{id}) \circ \psi_2 \circ (\mathrm{id} \otimes \tilde{\mathrm{pr}}_2)\\ &= (\pi_1 \otimes \mathrm{id}) \circ \psi_1 \circ (\mathrm{id} \otimes \tilde{\mathrm{pr}}_1) - (\pi_2 \otimes \mathrm{id}) \circ \psi_2 \circ (\mathrm{id} \otimes \tilde{\mathrm{pr}}_2)\\ &= \psi_{12} \circ (\mathrm{id} \otimes \pi_1) \circ (\mathrm{id} \otimes \tilde{\mathrm{pr}}_1) - \psi_{\pi_2(P_2)} \circ (\mathrm{id} \otimes \pi_2) \circ (\mathrm{id} \otimes \tilde{\mathrm{pr}}_2)\\ &= \psi_{12} \circ \big(\mathrm{id} \otimes (\pi_1 \circ \tilde{\mathrm{pr}}_1 - \pi_2 \circ \tilde{\mathrm{pr}}_2)\big).\end{aligned} \tag{2.30}$$

Hence $\tilde{\psi}(C \otimes P) \subseteq P \otimes C$. Much the same way, using (2.14) instead of (2.12), we show that the bijection $\tilde{\psi}^{-1}(P \otimes C) \subseteq C \otimes P$.

It remains to verify that the bijection $\psi = \tilde{\psi}\restriction_{C\otimes P}$ is an entwining that makes P an entwined module. The former is proven by a direct checking of (1.15) and (1.16). The latter follows from the fact that P_1 and P_2 are, respectively, ψ_1 and ψ_2 entwined modules:

$$\begin{aligned}
\Delta_P(pq) &= \Delta_{P_1}(\mathrm{pr}_1(p)\mathrm{pr}_1(q)) + \Delta_{P_2}(\mathrm{pr}_2(p)\mathrm{pr}_2(q)) \\
&= \mathrm{pr}_1(p_{(0)})\psi_1(p_{(1)} \otimes \mathrm{pr}_1(q)) + \mathrm{pr}_2(p_{(0)})\psi_2(p_{(1)} \otimes \mathrm{pr}_2(q)) \\
&= \big(\mathrm{pr}_1(p_{(0)}) + \mathrm{pr}_2(p_{(0)})\big)\big(\psi_1(p_{(1)} \otimes \mathrm{pr}_1(q)) + \psi_2(p_{(1)} \otimes \mathrm{pr}_2(q))\big) \\
&= p_{(0)}\psi(p_{(1)} \otimes q). \qquad (2.31)
\end{aligned}$$

This proves the lemma. $\square$

Let α_L^1 and α_R^1 be a unital left colinear splitting and a unital right colinear splitting of π_1, respectively. Also, let α_R^2 be a right colinear splitting of π_2 viewed as a map onto $\pi_2(P_2)$. Such maps exist by Lemma 2.1. On the other hand, by [9, Lemma 2.2], since P_1 and P_2 are principal, they admit strong connections ℓ_1 and ℓ_2, respectively. For brevity, let us introduce the notation

$$\alpha_L^{12} := \alpha_L^1\circ\pi_2, \quad \alpha_R^{12} := \alpha_R^1\circ\pi_2, \quad \alpha_R^{21} := \alpha_R^2\circ\pi_1\restriction_{\pi_1^{-1}(\pi_2(P_2))}, \quad L := m_{P_1}\circ(\alpha_L^{12}\otimes\alpha_R^{12})\circ\ell_2\,, \quad (2.32)$$

where m_{P_1} is the multiplication of P_1. The situation is illustrated in the following diagram:

$$\begin{tikzcd}
C \arrow[dd, "L"'] & P \arrow[ddl, "\mathrm{pr}_1"'] \arrow[ddr, "\mathrm{pr}_2"] & \\
& \pi_1^{-1}(\pi_2(P_2)) \arrow[dl, hook'] \arrow[dr, "\alpha_R^{21}"] & \\
P_1 \arrow[ddr, "\pi_1"] & & P_2 \arrow[ll, shift right, "\alpha_L^{12}"'] \arrow[ll, shift left, "\alpha_R^{12}"] \arrow[dl, shift left, "\pi_2"] \arrow[ddl, "\pi_2"] \\
& \pi_2(P_2) \arrow[ur, shift left, "\alpha_R^2"] \arrow[d, hook] & \\
& P_{12} \arrow[uul, shift left, "\alpha_R^1"] \arrow[uul, shift right, "\alpha_L^1"'] &
\end{tikzcd} \qquad (2.33)$$

Our proves hinges on constructing a strong connection on P out of strong connections on P_1 and P_2. As a first approximation for constructing a strong connection on P, we choose the formula

$$\ell_I := \big((\alpha_L^{12} + \mathrm{id}) \otimes (\alpha_R^{12} + \mathrm{id})\big) \circ \ell_2. \qquad (2.34)$$

It is evidently a bicolinear map fom C to $P \otimes P$ satisfying $\ell_I(e) = 1 \otimes 1$. However, it does not split the lifted canonical map:

$$\begin{aligned}
&(\widetilde{\mathsf{can}} \circ \ell_I)(c) - 1 \otimes c \\
&\quad = \alpha_L^{12}(\ell_2(c)^{\langle 1\rangle})\alpha_R^{12}(\ell_2(c)^{\langle 2\rangle})_{(0)} \otimes \alpha_R^{12}(\ell_2(c)^{\langle 2\rangle})_{(1)} + \ell_2(c)^{\langle 1\rangle}\ell_2(c)^{\langle 2\rangle}{}_{(0)} \otimes \ell_2(c)^{\langle 2\rangle}{}_{(1)} - 1 \otimes c \\
&\quad = \alpha_L^{12}(\ell_2(c_{(1)})^{\langle 1\rangle})\alpha_R^{12}(\ell_2(c_{(1)})^{\langle 2\rangle}) \otimes c_{(2)} + (0,1) \otimes c - 1 \otimes c \\
&\quad = L(c_{(1)}) \otimes c_{(2)} - (1,0) \otimes c \ \in \ P_1 \otimes C. \qquad (2.35)
\end{aligned}$$

Therefore, we correct it by adding to $\ell_I(c)$ the splitting of the lifted canonical map on $P_1\otimes P_1$ afforded by ℓ_1 and applied to $(1,0)\otimes c - L(c_{(1)})\otimes c_{(2)}$:

$$\begin{aligned}\ell_{II}(c) &= \ell_I(c) + \ell_1(c)^{\langle 1\rangle}\otimes \ell_1(c)^{\langle 2\rangle} - L(c_{(1)})\,\ell_1(c_{(2)})^{\langle 1\rangle}\otimes \ell_1(c_{(2)})^{\langle 2\rangle}\\ &= (\ell_I + \ell_1 - L*\ell_1)(c).\end{aligned} \tag{2.36}$$

The above approximation to a strong connection on P is clearly right colinear. Using the fact that P_1 is a ψ_1-entwined and e-coaugmented module, we follow the lines of (2.21)–(2.23) to show that $L*\ell_1$ is left colinear. Hence ℓ_{II} is bicolinear. It also satisfies $\ell_{II}(e)=1\otimes 1$. However, the price we pay for having $\ell_{II}(c)^{\langle 1\rangle}\,\ell_{II}(c)^{\langle 2\rangle}{}_{(0)}\otimes \ell_{II}(c)^{\langle 2\rangle}{}_{(1)} = 1\otimes c$ is that the image of ℓ_{II} is no longer in $P\otimes P$.

The troublesome term $\ell_1 - L*\ell_1$ takes values in $P\otimes(P_1\times P_2)$. Its right-sided version $\ell_1-\ell_1*L$ takes values in $(P_1\times P_2)\otimes P$. Plugging one into another yields a map $\ell_1 - L*\ell_1 - \ell_1*L + L*\ell_1*L$ who's image is in $P\otimes P$. Thus our third approximation is:

$$\ell_{III} = \ell_{II} - \ell_1 * L + L*\ell_1*L. \tag{2.37}$$

The bicolinearity can be shown using again properties of entwined coaugmented modules. (See the above argument for the left colinearity of $L*\ell_1$.) The property $\ell_{III}(e)=1\otimes 1$ is evident. However, enforcing $\ell_{III}(c)\subseteq P\otimes P$ spoiled the splitting property:

$$\begin{aligned}\widetilde{\mathsf{can}}(\ell_{III}(c)) &= 1\otimes c - \ell_1(c_{(1)})^{\langle 1\rangle}\Delta_{P_1}\big(\ell_1(c_{(1)})^{\langle 2\rangle}L(c_{(2)})\big) + L(c_{(1)})\,\ell_1(c_{(2)})^{\langle 1\rangle}\Delta_{P_1}\big(\ell_1(c_{(2)})^{\langle 2\rangle}L(c_{(3)})\big)\\ &= 1\otimes c - \ell_1(c_{(1)})^{\langle 1\rangle}\ell_1(c_{(1)})^{\langle 2\rangle}\psi_1\big(c_{(2)}\otimes L(c_{(3)})\big) + L(c_{(1)})\,\ell_1(c_{(2)})^{\langle 1\rangle}\ell_1(c_{(2)})^{\langle 2\rangle}\psi_1\big(c_{(3)}\otimes L(c_{(4)})\big)\\ &= 1\otimes c - \psi_1\big(c_{(1)}\otimes L(c_{(2)})\big) + L(c_{(1)})\,\psi_1\big(c_{(2)}\otimes L(c_{(3)})\big)\end{aligned} \tag{2.38}$$

Finally, it follows from Lemma 2.1 that we can always choose a strong connection ℓ_1 satisfying $\ell_1(C)\subseteq P_1\otimes \pi_1^{-1}(\pi_2(P_2))$. Now we remedy the situation by replacing the troublesome term $L*\ell_1*L-\ell_1*L$ with $-(\mathrm{id}\otimes\alpha_R^{21})\circ(L*\ell_1*L-\ell_1*L)$. Indeed, since

$$(\mathrm{id}\otimes\pi_1)(L*\ell_1*L-\ell_1*L)(c) = (L-\varepsilon)(c_{(1)})\,\ell(c_{(2)})^{\langle 1\rangle}\otimes \pi_1(\ell_1(c_{(2)})^{\langle 2\rangle}) \in P_1\otimes\pi_2(P_2), \tag{2.39}$$

the map $(\mathrm{id}\otimes\alpha_R^{21})\circ(L*\ell_1*L-\ell_1*L)$ is well defined. Furthermore, it is evidently annihilated by the lifted canonical map. Hence

$$\ell_{IV} := \ell_{II} - (\mathrm{id}\otimes\alpha_R^{21})\circ(L*\ell_1*L-\ell_1*L) \tag{2.40}$$

is a bicolinear map satisfying $\ell_{IV}(e)=1\otimes 1$ and $\widetilde{\mathsf{can}}\circ\ell_{IV} = 1\otimes\mathrm{id}$. Moreover, this time it also enjoys the property $\ell_{IV}(C)\subseteq P\otimes P$:

$$\begin{aligned}&\big((\pi_1\circ \mathrm{pr}_1 - \pi_2\circ\mathrm{pr}_2)\otimes\mathrm{id}\big)\circ(\ell_{III}-\ell_{IV})\\ &\quad= \big((\pi_1\circ \mathrm{pr}_1 - \pi_2\circ\mathrm{pr}_2)\otimes\mathrm{id}\big)\circ\big(\mathrm{id}\otimes(\mathrm{id}+\alpha_R^{21})\big)\circ(L*\ell_1*L-\ell_1*L)\\ &\quad= \big(\mathrm{id}\otimes(\mathrm{id}+\alpha_R^{21})\big)\circ\big((\pi_1\circ \mathrm{pr}_1 - \pi_2\circ\mathrm{pr}_2)\otimes\mathrm{id}\big)\circ(L*\ell_1*L-\ell_1*L)\\ &\quad= 0,\end{aligned} \tag{2.41}$$

$$\begin{aligned}&\big(\mathrm{id}\otimes(\pi_1\circ \mathrm{pr}_1 - \pi_2\circ\mathrm{pr}_2)\big)\circ(\ell_{III}-\ell_{IV})\\ &\quad= \big(\mathrm{id}\otimes(\pi_1\circ \mathrm{pr}_1 - \pi_2\circ\mathrm{pr}_2)\circ(\mathrm{id}+\alpha_R^{21})\big)\circ(L*\ell_1*L-\ell_1*L)\\ &\quad= \big(\mathrm{id}\otimes(\pi_1-\pi_1)\big)\circ(L*\ell_1*L-\ell_1*L)\\ &\quad= 0.\end{aligned} \tag{2.42}$$

Hence ℓ_{IV} is a desired strong connection on P. Combining this fact with Lemma 2.3 and Lemma 1.2 proves the theorem. □

Putting the formulas in the proof of Theorem 2.2 together, we obtain the following strong connection on P:

$$\begin{aligned}\ell \ &= \big((\alpha_L^{12} + \mathrm{id}) \otimes (\alpha_R^{12} + \mathrm{id})\big) \circ \ell_2 \\ &\quad + \ (\eta_1 \circ \varepsilon - L) * \Big(\big(\mathrm{id} \otimes (\mathrm{id} + \alpha_R^{21})\big) \circ \big(\ell_1 - \ell_1 * L + (\alpha_L^{12} \otimes \alpha_R^{12}) \circ \ell_2\big)\Big).\end{aligned} \tag{2.43}$$

3 The pullback picture of the standard quantum Hopf fibration

3.1 Pullback comodule algebra

Recall that $H := \mathcal{O}(\mathrm{U}(1))$ is the commutative and cocommutative Hopf *-algebra generated by a grouplike unitary element v. This means that v satisfies the relations $vv^* = v^*v = 1$ and $\Delta(v) = v \otimes v$. As a consequence, the counit ε and the antipode S yield $\varepsilon(v) = 1$ and $S(v) = v^*$.

Consider the complex tensor products $A_1 := \mathcal{T} \otimes \mathcal{O}(\mathrm{U}(1))$, $A_2 = \mathbb{C} \otimes \mathcal{O}(\mathrm{U}(1)) \cong \mathcal{O}(\mathrm{U}(1))$ and $A_{12} := \mathcal{C}(\mathrm{S}^1) \otimes \mathcal{O}(\mathrm{U}(1))$. These algebras are right H-comodule algebras with the trivial right H-coaction $x \otimes v^N \mapsto x \otimes v^N \otimes v^N$, $N \in \mathbb{Z}$. Moreover, A_1 and A_2 are trivially principal with strong connections $\ell_i : H \to A_i \otimes A_i$, $i=1,2$, given by $\ell_i(v^N) = (1 \otimes v^{N*}) \otimes (1 \otimes v^N)$.

We define morphisms of right H-comodule algebras by

$$\pi_1 : \mathcal{T} \otimes \mathcal{O}(\mathrm{U}(1)) \longrightarrow \mathcal{C}(\mathrm{S}^1) \otimes \mathcal{O}(\mathrm{U}(1)), \quad \pi_1(t \otimes v^N) = \sigma(t) \otimes v^N, \tag{3.1}$$

$$\pi_2 : \mathcal{O}(\mathrm{U}(1)) \longrightarrow \mathcal{C}(\mathrm{S}^1) \otimes \mathcal{O}(\mathrm{U}(1)), \quad \pi_2(v^N) = 1 \otimes v^N, \tag{3.2}$$

$$\Phi : \mathcal{C}(\mathrm{S}^1) \otimes \mathcal{O}(\mathrm{U}(1)) \longrightarrow \mathcal{C}(\mathrm{S}^1) \otimes \mathcal{O}(\mathrm{U}(1)), \quad \Phi(f \otimes v^N) = fu^N \otimes v^N. \tag{3.3}$$

Then the fibre product $P := \mathcal{T} \otimes \mathcal{O}(\mathrm{U}(1)) \times_{(\Phi \circ \pi_1, \pi_2)} \mathcal{O}(\mathrm{U}(1))$ defined by the pullback diagram

$$\begin{tikzcd}[column sep=small] & \mathcal{T} \otimes \mathcal{O}(\mathrm{U}(1)) \underset{(\Phi\circ\pi_1,\pi_2)}{\times} \mathcal{O}(\mathrm{U}(1)) \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ \mathcal{T} \otimes \mathcal{O}(\mathrm{U}(1)) \arrow[d, "t\otimes v^N \overset{\pi_1}{\longmapsto} \sigma(t)\otimes v^N"'] & & \mathcal{O}(\mathrm{U}(1)) \arrow[d, "v^N \overset{\pi_2}{\longmapsto} 1\otimes v^N"] \\ \mathcal{C}(\mathrm{S}^1) \otimes \mathcal{O}(\mathrm{U}(1)) \arrow[rr, "f\otimes v^N \overset{\Phi}{\longmapsto} fu^N\otimes v^N"'] & & \mathcal{C}(\mathrm{S}^1) \otimes \mathcal{O}(\mathrm{U}(1)) \end{tikzcd} \tag{3.4}$$

is a right $\mathcal{O}(\mathrm{U}(1))$-comodule algebra. By Proposition 2.2, it is principal. For $N \in \mathbb{N}$, lifting $1 \otimes v^N$ and $1 \otimes v^{*N}$ in $\mathcal{C}(\mathrm{S}^1) \otimes \mathcal{O}(\mathrm{U}(1))$ to $s^{*N} \otimes v^N$ and $s^N \otimes v^{*N}$ in $\mathcal{T} \otimes \mathcal{O}(\mathrm{U}(1))$, respectively, we can define a strong connection by a formula analogous to (2.43):

$$\begin{aligned}\ell(u^N) &= (s^N \otimes v^{*N}, v^{*N}) \otimes (s^{*N} \otimes v^N, v^N) \\ &\qquad + ((1 - s^N s^{*N}) \otimes v^{*N}, 0) \otimes ((1 - s^N s^{*N}) \otimes v^N, 0),\end{aligned} \tag{3.5}$$

$$\ell(u^{*N}) = (s^{*N} \otimes v^N, v^N) \otimes (s^N \otimes v^{*N}, v^{*N}). \tag{3.6}$$

One can check now directly that ℓ yields a strong connection.

By construction, we have

$$P = \{\textstyle\sum_k (t_k \otimes v^k, \alpha_k v^k) \in \mathcal{T} \otimes \mathcal{O}(\mathrm{U}(1)) \times \mathcal{O}(\mathrm{U}(1)) \mid \sigma(t_k)u^k = \alpha_k\}, \tag{3.7}$$

where $\alpha_k \in \mathbb{C}$. Set

$$L_N := \{p \in P \mid \Delta_P(p) = p \otimes v^N\}. \tag{3.8}$$

Then $L_0 = P^{\mathrm{co}\mathcal{O}(\mathrm{U}(1))}$, each L_N is a left $P^{\mathrm{co}\mathcal{O}(\mathrm{U}(1))}$-module and $P = \bigoplus_{N\in\mathbb{Z}} L_N$. From

$$\Delta_P(\sum_k (t_k \otimes v^k, \alpha_k v^k)) = \sum_k (t_k \otimes v^k, \alpha_k v^k) \otimes v^k, \tag{3.9}$$

it follows that

$$L_N = \{(t \otimes v^N, \alpha v^N) \in \mathcal{T} \otimes \mathcal{O}(\mathrm{U}(1)) \times \mathcal{O}(\mathrm{U}(1)) \mid \sigma(t)u^N = \alpha \in \mathbb{C}\}. \tag{3.10}$$

Hence $L_N \cong \mathcal{T} \times_{(u^N\sigma,1)} \mathcal{O}(\mathrm{U}(1))$, where $\mathcal{T} \times_{(u^N\sigma,1)} \mathcal{O}(\mathrm{U}(1))$ is given by the pullback diagram

$$\begin{tikzcd}
 & \mathcal{T} \underset{(u^N\sigma,1)}{\times} \mathbb{C} \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\
\mathcal{T} \arrow[d, "\sigma"'] & & \mathbb{C} \arrow[d, "\alpha\mapsto\alpha 1"] \\
\mathcal{C}(\mathrm{S}^1) \arrow[rr, "f\mapsto u^N f"'] & & \mathcal{C}(\mathrm{S}^1).
\end{tikzcd} \tag{3.11}$$

The next proposition shows that $L_0 \cong \mathcal{T} \times_{(\sigma,1)} \mathcal{O}(\mathrm{U}(1))$ is isomorphic to the C^*-algebra of the standard Podleś sphere and that $\mathcal{T} \times_{(u^N\sigma,1)} \mathcal{O}(\mathrm{U}(1))$ describes the associated line bundles.

Proposition 3.1. *The fibre product $\mathcal{T}\times_{(\sigma,1)}\mathbb{C}$ is isomorphic to the C^*-algebra $\mathcal{C}(\mathrm{S}^2_q)$, and $L_N \cong \mathcal{T}\times_{(u^N\sigma,1)} \mathcal{O}(\mathrm{U}(1))$ is an isomorphism of left $\mathcal{C}(\mathrm{S}^2_q)$-modules with respect to the left $\mathcal{C}(\mathrm{S}^2_q)$-action on $\mathcal{T} \times_{(u^N\sigma,1)} \mathcal{O}(\mathrm{U}(1))$ given by $(t,\alpha).(h,\beta) = (th, \alpha\beta)$.*

Proof. For $N = 0$, the mappings $\mathcal{T} \ni t \mapsto \sigma(t) \in \mathcal{C}(\mathrm{S}^1)$ and $\mathbb{C} \ni \alpha \mapsto \alpha 1 \in \mathcal{C}(\mathrm{S}^1)$ are morphisms of C^*-algebras, thus $\mathcal{T}\times_{(\sigma,1)}\mathbb{C}$ is a C^*-algebra.

Recall that $\mathcal{C}(\mathrm{S}^2_q) \cong \mathcal{K}(\ell_2(\mathbb{N})) \oplus \mathbb{C}$. Define

$$\psi : \mathcal{T} \underset{(\sigma,1)}{\times} \mathbb{C} \longrightarrow \mathcal{K}(\ell_2(\mathbb{N})) \oplus \mathbb{C}, \quad \psi(t,\alpha) = t, \tag{3.12}$$

$$\phi : \mathcal{K}(\ell_2(\mathbb{N})) \oplus \mathbb{C} \longrightarrow \mathcal{T} \underset{(\sigma,1)}{\times} \mathbb{C}, \quad \phi(k+\alpha) = (k+\alpha, \alpha). \tag{3.13}$$

Clearly, $\psi : \mathcal{T}\times_{(\sigma,1)}\mathbb{C} \to \mathcal{B}(\ell_2(\mathbb{N}))$ is a morphism of C^*-algebras. Since $\psi(t,\alpha) = (t-\alpha)+\alpha$, and $\sigma(t-\alpha) = 0$ by the pullback diagram (3.11), it follows from the short exact sequence (1.60) that $t-\alpha \in \mathcal{K}(\ell_2(\mathbb{N}))$, so $\psi(t,\alpha) \in \mathcal{K}(\ell_2(\mathbb{N})) \oplus \mathbb{C}$. Using $kh + \alpha h + \beta k \in \mathcal{K}(\ell_2(\mathbb{N}))$ for all $k, h \in \mathcal{K}(\ell_2(\mathbb{N}))$ and $\alpha, \beta \in \mathbb{C}$, one easily sees that ϕ is also a morphism of C^*-algebras.

Now, for all $(t,\alpha) \in \mathcal{T}\times_{(\sigma,1)}\mathbb{C}$ and $k+\alpha \in \mathcal{K}(\ell_2(\mathbb{N}))\oplus\mathbb{C}$, we get $\phi\circ\psi(t,\alpha) = \phi((t-\alpha)+\alpha) = (t,\alpha)$ and $\psi\circ\phi(k+\alpha) = \psi(k+\alpha,\alpha) = k+\alpha$, so that $\mathcal{T}\times_{(\sigma,1)}\mathbb{C} \cong \mathcal{K}(\ell_2(\mathbb{N})) \oplus \mathbb{C}$.

The fact that $\mathcal{T} \times_{(u^N\sigma,1)} \mathbb{C}$ with the given $\mathcal{C}(\mathrm{S}^2_q)$-action is a left $\mathcal{C}(\mathrm{S}^2_q)$-module follows from the discussion preceding the pullback diagram (1.9) with the free rank 1 modules $E_1 = \mathcal{T}$, $E_2 = \mathbb{C}$ and $\pi_{1*}E_1 = \pi_{2*}E_2 = \mathcal{C}(\mathrm{S}^1)$. Obviously, $L_N \ni (t \otimes v^N, \alpha v^N) \mapsto (t, \alpha) \in \mathcal{T} \times_{(u^N\sigma,1)} \mathbb{C}$ defines an isomorphism of left $\mathcal{C}(\mathrm{S}^2_q)$-modules. $\square$

Classically, one can construct complex line bundles over the 2-sphere by glueing two trivial line bundles over the unit disc along their boundaries. If we first rotate one of the trivial line bundles at each point $\mathrm{e}^{\mathrm{i}\phi}$ of the boundary by the phase $\mathrm{e}^{\mathrm{i}N\phi} \in \mathrm{U}(1)$, where $N \in \mathbb{Z}$, and then glue it to the other one, we obtain a line bundle over the 2-sphere with the winding number N. Topologically, the "un-rotated" disc can be shrinked to a point. The pullback diagram (3.11) can be viewed as a quantum analogue of this construction.

3.2 The equivalence of the pullback and standard constructions

Let $\bar{H} := \mathcal{C}(\mathrm{S}^1)$. It is trivially a compact quantum group with the Hopf algebra structure induced by the group operations of S^1, that is, $\Delta(f)(x,y) = f(xy)$, $\varepsilon(f) = f(1)$ and $S(f)(x) = f(x^{-1})$. Its dense Hopf *-subalgebra spanned by the matrix coefficients of the irreducible unitary corepresentations is given by $H = \mathcal{O}(\mathrm{U}(1))$ with generator $u \in \mathcal{C}(\mathrm{S}^1)$, $u(\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{\mathrm{i}\theta}$.

Define a C^*-algebra morphism $W : \mathcal{C}(\mathrm{S}^1)\bar{\otimes}\mathcal{C}(\mathrm{S}^1) \to \mathcal{C}(\mathrm{S}^1)\bar{\otimes}\mathcal{C}(\mathrm{S}^1)$ by $W(f)(x,y) = f(x,xy)$ (known as multiplicative unitary). Let $\pi_2 : \mathcal{C}(\mathrm{S}^1) \to \mathcal{C}(\mathrm{S}^1)\bar{\otimes}\mathcal{C}(\mathrm{S}^1)$ be given by $\pi_2(f)(x,y) = f(y)$ and let $\sigma\bar{\otimes}\mathrm{id}$ denote the tensor product of the C^*-algebra morphisms $\sigma : \mathcal{T} \to \mathcal{C}(\mathrm{S}^1)$ and $\mathrm{id} : \mathcal{C}(\mathrm{S}^1) \to \mathcal{C}(\mathrm{S}^1)$. Then $\bar{P} := \mathcal{T}\bar{\otimes}\mathcal{C}(\mathrm{S}^1) \times_{(W\circ\sigma\bar{\otimes}\mathrm{id},\pi_2)} \mathcal{C}(\mathrm{S}^1)$ is defined by the pullback diagram

$$
\begin{tikzcd}
 & \mathcal{T}\bar{\otimes}\mathcal{C}(\mathrm{S}^1) \underset{(W\circ\sigma\bar{\otimes}\mathrm{id},\pi_2)}{\times} \mathcal{C}(\mathrm{S}^1) \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\
\mathcal{T}\bar{\otimes}\mathcal{C}(\mathrm{S}^1) \arrow[d, "\sigma\bar{\otimes}\mathrm{id}"'] & & \mathcal{C}(\mathrm{S}^1) \arrow[d, "\pi_2"] \\
\mathcal{C}(\mathrm{S}^1)\bar{\otimes}\mathcal{C}(\mathrm{S}^1) \arrow[rr, "W"'] & & \mathcal{C}(\mathrm{S}^1)\bar{\otimes}\mathcal{C}(\mathrm{S}^1) .
\end{tikzcd}
\tag{3.14}
$$

With the $\bar{H}$-coaction given by the coproduct Δ on the (right) tensor factor $\bar{H} = \mathcal{C}(\mathrm{S}^1)$, the mappings $\sigma\bar{\otimes}\mathrm{id}$, π_2 and W are morphisms in the category of right $\bar{H}$-comodule C^*-algebras. Therefore $\bar{P}$ inherits the structure of a right $\bar{H}$-comodule C^*-algebra.

In Section 1.3, the Peter-Weyl comodule algebra was defined by those elements of $\bar{P}$ for which the right $\bar{H}$-coaction lands in $\bar{P} \otimes H$, where, in our case, $H = \mathcal{O}(\mathrm{U}(1))$ is the dense Hopf *-subalgebra of $\bar{H}$ spanned by the matrix coefficients of the irreducible unitary corepresentations. Since the right $\bar{H}$-coaction is given by the coproduct on $\bar{H} = \mathcal{C}(\mathrm{S}^1)$, it follows that only the coaction of elements contained in $\mathcal{T} \otimes \mathcal{O}(\mathrm{U}(1)) \times \mathcal{O}(\mathrm{U}(1))$ lands in $\bar{P} \otimes H$. Moreover, for $f \otimes u^N \in \mathcal{C}(\mathrm{S}^1) \otimes \mathcal{O}(\mathrm{U}(1))$, we have $W(f \otimes u^N) = fu^N \otimes u^N$ which coincides with the mapping Φ defined in the previous section. Therefore the Peter-Weyl comodule algebra is isomorphic to fibre product $P = \mathcal{T} \otimes \mathcal{O}(\mathrm{U}(1)) \times_{(\Phi\circ\pi_1,\pi_2)} \mathcal{O}(\mathrm{U}(1))$ of Section 3.1.

Consider now the *-representation of $\mathcal{O}(\mathrm{SU}_q(2))$ on $\ell_2(\mathbb{N})$ given by

$$\begin{aligned} \rho(\alpha)e_n &= (1-q^{2n})^{1/2}e_{n-1}, \quad \rho(\beta)e_n = -q^{n+1}e_n, \\ \rho(\gamma)e_n &= q^n e_n, \quad \rho(\delta)e_n = (1-q^{2(n+1)})^{1/2}e_{n+1}. \end{aligned} \tag{3.15}$$

Note that $\rho(\beta), \mu(\gamma) \in \mathcal{K}(\ell_2(\mathbb{N}))$. Comparing ρ with the representation μ of $\mathcal{O}(\mathrm{D}_q)$, one readily sees that $\rho(\mathcal{O}(\mathrm{SU}_q(2))) \subseteq \mathcal{T}$. Moreover, the symbol map σ yields $\sigma(\rho(\alpha)) = u^{-1}$, $\sigma(\rho(\delta)) = u$ and $\sigma(\rho(\beta)) = \sigma(\rho(\gamma)) = 0$. Therefore, we obtain a morphism of $\mathcal{O}(\mathrm{U}(1))$-comodule *-algebras $\iota : \mathcal{O}(\mathrm{SU}_q(2)) \to P$ by setting

$$\iota(\alpha) = (\rho(\alpha)\otimes v, v), \quad \iota(\gamma) = (\rho(\gamma)\otimes v, 0). \tag{3.16}$$

One easily checks that the image of a Poincaré-Birkhoff-Witt basis of $\mathcal{O}(\mathrm{SU}_q(2))$ remains linearly independent, so ι is injective and we can consider $\mathcal{O}(\mathrm{SU}_q(2))$ as a subalgebra of P. In particular, we have $M_{-N} \subseteq L_N$ as left $\mathcal{O}(\mathrm{S}_q^2)$-modules. (See Section 1.4 and Section 3.1 for the definitions of M_N and L_N, respectively.)

The main objective of this section is to establish an $\mathcal{C}(\mathrm{S}^1)$-comodule C^*-algebra isomorphism $\mathcal{C}(\mathrm{SU}_q(2)) \cong \bar{P}$. The universal C^*-algebra $\mathcal{C}(\mathrm{SU}_q(2))$ of $\mathcal{O}(\mathrm{SU}_q(2))$ has been studied in [19] and [30]. Here we shall use the fact from [19, Corollary 2.3] that $\mathcal{C}(\mathrm{SU}_q(2))$ is generated by the operators $\hat{\alpha}$ and $\hat{\gamma}$ acting on the Hilbert space $\ell_2(\mathbb{N})\bar{\otimes}\ell_2(\mathbb{Z})$ by

$$\hat{\alpha}(e_n\otimes f_k) = (1-q^{2n})^{1/2}e_{n-1}\otimes f_k, \quad \hat{\gamma}(e_n\otimes f_k) = q^n e_n\otimes f_{k-1}, \tag{3.17}$$

where $\{e_n\}_{n\in\mathbb{N}}$ and $\{f_k\}_{k\in\mathbb{Z}}$ denote the standard bases of $\ell_2(\mathbb{N})$ and $\ell_2(\mathbb{Z})$, respectively. The right $\mathcal{C}(\mathrm{S}^1)$-coaction on $\mathcal{C}(\mathrm{SU}_q(2))$ is given by $(\mathrm{id}\bar{\otimes}\bar{\pi})\circ\Delta$, where Δ denotes the coproduct of the compact quantum group $\mathcal{C}(\mathrm{SU}_q(2))$ and $\bar{\pi}$ is the extension of the Hopf *-algebra surjection $\pi : \mathcal{O}(\mathrm{SU}_q(2)) \to \mathcal{O}(\mathrm{U}(1))$ from Section 1.4 to the C^*-closures.

Theorem 3.2. *The $\mathcal{C}(\mathrm{S}^1)$-comodule C^*-algebras $\mathcal{C}(\mathrm{SU}_q(2))$ and $\bar{P}$ are isomorphic.*

Proof. First we realize $\mathcal{C}(\mathrm{S}^1)$ as a concrete C^*-algebra of bounded operators on $\ell_2(\mathbb{Z})$ by setting $v(f_k) = f_{k-1}$. After applying the unitary transformation $T : \ell_2(\mathbb{N})\bar{\otimes}\ell_2(\mathbb{Z}) \to \ell_2(\mathbb{N})\bar{\otimes}\ell_2(\mathbb{Z})$, $T(e_n\otimes f_k) = e_n\otimes f_{k-n}$, we obtain from (3.17) a concrete realization $\hat{\rho}$ of $\mathcal{C}(\mathrm{SU}_q(2))$ on $\ell_2(\mathbb{N})\bar{\otimes}\ell_2(\mathbb{Z})$ such that the generators are given by

$$\hat{\rho}(\alpha) = T^{-1}\hat{\alpha}T = \rho(\alpha)\otimes v, \quad \hat{\rho}(\gamma) = T^{-1}\hat{\gamma}T = \rho(\gamma)\otimes v. \tag{3.18}$$

Identifying $\mathcal{C}(\mathrm{S}^1)$ with its concrete realization on $\ell_2(\mathbb{Z})$, we get an obvious *-representation of the C^*-algebra $\bar{P} = \mathcal{T}\bar{\otimes}\mathcal{C}(\mathrm{S}^1)\times_{(W\circ\sigma\bar{\otimes}\mathrm{id},\pi_2)}\mathcal{C}(\mathrm{S}^1)$ on $\ell_2(\mathbb{N})\bar{\otimes}\ell_2(\mathbb{Z})$ by taking the left projection $\mathrm{pr}_1(x,y) = x$. Now, consider the C^*-algebra morphism $\varepsilon_1 : \mathcal{C}(\mathrm{S}^1)\bar{\otimes}\mathcal{C}(\mathrm{S}^1) \to \mathcal{C}(\mathrm{S}^1)$ given by $\varepsilon_1(f)(y) = f(1,y)$. From the pullback diagram (3.14), it follows that $\varepsilon_1\circ W\circ(\sigma\bar{\otimes}\mathrm{id})(x) = y$ for all $(x,y)\in\mathcal{T}\bar{\otimes}\mathcal{C}(\mathrm{S}^1)\times_{(W\circ\sigma\bar{\otimes}\mathrm{id},\pi_2)}\mathcal{C}(\mathrm{S}^1)$. Since C^*-algebra morphisms do not increase the norm, it follows that $||x|| \geq ||y||$, whence $||(x,y)|| = ||x||$. As a consequence, the mapping pr_1 is an isometry, so that we can identify $\bar{P}$ with its image under pr_1. By Equations (3.16) and (3.18), $\mathrm{pr}_1\circ\iota(x) = \hat{\rho}(x)$ for all $x\in\mathcal{O}(\mathrm{SU}_q(2))$. This implies that $\mathcal{C}(\mathrm{SU}_q(2)) \subseteq \mathrm{pr}_1(\bar{P}) \cong \bar{P}$.

It now suffices to prove that the image of $\mathcal{O}(\mathrm{SU}_q(2))$ under ι is dense in $\bar{P}$. Let $P_0 := \mathcal{O}(\mathrm{D}_q) \otimes \mathcal{O}(\mathrm{U}(1)) \times_{(\Phi\circ\pi_1,\pi_2)} \mathcal{O}(\mathrm{U}(1))$ be given by the pullback diagram

$$\begin{tikzcd}
& \mathcal{O}(\mathrm{D}_q) \otimes \mathcal{O}(\mathrm{U}(1)) \underset{(\Phi\circ\pi_1,\pi_2)}{\times} \mathcal{O}(\mathrm{U}(1)) \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\
\mathcal{O}(\mathrm{D}_q) \otimes \mathcal{O}(\mathrm{U}(1)) \arrow[d, "t\otimes v^N \overset{\pi_1}{\longmapsto} \sigma(t)\otimes v^N"'] & & \mathcal{O}(\mathrm{U}(1)) \arrow[d, "v^N \overset{\pi_2}{\longmapsto} 1\otimes v^N"] \\
\mathcal{O}(\mathrm{U}(1)) \otimes \mathcal{O}(\mathrm{U}(1)) \arrow[rr, "f\otimes v^N \overset{\Phi}{\longmapsto} fu^N\otimes v^N"'] & & \mathcal{O}(\mathrm{U}(1)) \otimes \mathcal{O}(\mathrm{U}(1))
\end{tikzcd} \tag{3.19}$$

Comparing the definitions of P_0, P and $\bar{P}$ shows that $P_0 \subseteq P \subseteq \bar{P}$. Since $\mathcal{O}(\mathrm{D}_q)$ is dense in $\mathcal{T}$ and $\mathcal{O}(\mathrm{U}(1))$ is dense in $\mathcal{C}(\mathrm{S}^1)$, we conclude from Theorem 1.1 and Lemma 1.4 that P_0 is dense in $\bar{P}$. Let z and z^* denote the generators of $\mathcal{O}(\mathrm{D}_q)$. Viewed as elements in $\mathcal{T}$, we have $\sigma(z) = u$ and $\sigma(z^*) = u^{-1}$ (cf. Section 1.4). Hence each element in P_0 is a linear combination of $(z^n z^{*n+k} \otimes v^k, v^k)$ and $(z^{n+k} z^{*n} \otimes v^{-k}, v^{-k})$, $n, k \in \mathbb{N}$. Since $\rho(\alpha) = z^*$, we have $\iota(\alpha^{*n}\alpha^{n+k}) = (z^n z^{*n+k} \otimes v^k, v^k)$ and $\iota(\alpha^{*n+k}\alpha^n) = (z^{n+k} z^{*n} \otimes v^{-k}, v^{-k})$, whence $P_0 \subseteq \iota(\mathcal{O}(\mathrm{SU}_q(2))) \subseteq \mathcal{C}(\mathrm{SU}_q(2))$. As P_0 is dense in $\bar{P}$, it follows that $\mathcal{C}(\mathrm{SU}_q(2)) = \bar{P}$.

From the pullback diagram (3.14), it follows that the C^*-isomorphism pr_1 is a morphism of right $\mathcal{C}(\mathrm{S}^1)$-comodule algebras. Thus, to establish an $\mathcal{C}(\mathrm{S}^1)$-comodule C^*-algebra isomorphism between $\mathcal{C}(\mathrm{SU}_q(2))$ and $\bar{P}$, we only need to prove that $\hat{\rho} : \mathcal{C}(\mathrm{SU}_q(2)) \to \mathrm{pr}_1(\bar{P})$ is a morphism of $\mathcal{C}(\mathrm{S}^1)$-comodule algebras. This can easily be checked on the subalgebra $\mathcal{O}(\mathrm{SU}_q(2))$. Since the $\mathcal{C}(\mathrm{S}^1)$-coaction $(\mathrm{id}\bar{\otimes}\bar{\pi}) \circ \Delta$ is continuous on $\mathcal{C}(\mathrm{SU}_q(2))$, the claim follows. □

Corollary 3.3. *For $N \in \mathbb{Z}$, define*

$$\bar{M}_N := \{p \in \mathcal{C}(\mathrm{SU}_q(2)) \mid (\mathrm{id}\bar{\otimes}\bar{\pi}) \circ \Delta(p) = p \otimes u^{-N}\}. \tag{3.20}$$

Let L_N be given by Equation (3.8). Then the left $\mathcal{C}(\mathrm{S}^2_q)$-modules L_{-N}, $\bar{M}_N$ and $\mathcal{C}(\mathrm{S}^2_q)^{|N|+1}E_N$ are isomorphic, where E_N denotes the projection matrix of Equation (1.65).

Proof. Since P is the Peter-Weyl comodule algebra of $\bar{P}$, we have

$$L_N = \{p \in \bar{P} \mid \Delta_{\bar{P}}(p) = p \otimes u^N\}, \tag{3.21}$$

and the isomorphism $L_{-N} \cong \bar{M}_N$ follows now from Theorem 3.2.

For $j \in \frac{1}{2}\mathbb{Z}$, set $T_{2j} := (t^{|j|}_{-|j|,j}, \ldots, t^{|j|}_{|j|,j})^{\mathrm{t}}$. Then $E_{2j} = T_{2j}\, T^*_{2j}$ and $T^*_{2j}\, T_{2j} = 1$. Recall from Section 1.4 that $(\mathrm{id} \otimes \pi) \circ \Delta(t^{|j|}_{k,j}) = \Delta_R(t^{|j|}_{k,j}) = t^{|j|}_{k,j} \otimes u^{-2j}$. Hence $t^{|j|}_{k,j} \in \bar{M}_{2j}$ for all $k = -|j|, \ldots, |j|$. Define a $\mathcal{C}(\mathrm{S}^2_q)$-linear mapping $\chi : \mathcal{C}(\mathrm{S}^2_q)^{|2j|+1}E_{2j} \to M_{2j}$ by

$$\chi\big((a_{-|j|}, \ldots, a_{|j|})E_{2j}\big) := \textstyle\sum_k a_k t^{|j|}_{k,j}. \tag{3.22}$$

Suppose we are given $a_{-|j|}, \ldots, a_{|j|} \in \mathcal{C}(\mathrm{S}^2_q)$ such that $(a_{-|j|}, \ldots, a_{|j|})E_{2j} = 0$. Using $E_{2j} = T_{2j}\, T^*_{2j}$ and $T^*_{2j}\, T_{2j} = 1$, we get

$$0 = (a_{-|j|}, \ldots, a_{|j|})T_{2j}\, T^*_{2j}\, T_{2j} = \textstyle\sum_k a_k t^{|j|}_{k,j} = \chi\big((a_{-|j|}, \ldots, a_{|j|})E_{2j}\big), \tag{3.23}$$

so that χ is well defined. If $\sum_k a_k t^{|j|}_{k,j} = 0$, then

$$(a_{-|j|}, \dots, a_{|j|})E_{2j} = \big((a_{-|j|}, \dots, a_{|j|})T_{2j}\big)T^*_{2j} = (\textstyle\sum_k a_k t^{|j|}_{k,j})T^*_{2j} = 0, \tag{3.24}$$

so that χ is injective. Let $x \in \bar{M}_{2j}$. Since $(\mathrm{id}\bar{\otimes}\bar{\pi}) \circ \Delta(xt^{|j|*}_{k,j}) = xt^{|j|*}_{k,j} \otimes u^{-2j}u^{2j} = xt^{|j|*}_{k,j} \otimes 1$, we have $xt^{|j|*}_{k,j} \in \mathcal{C}(\mathrm{S}^2_q) = \mathcal{C}(\mathrm{SU}_q(2))^{\mathrm{co}\mathcal{C}(\mathrm{S}^1)}$ and

$$\chi\big((xt^{|j|*}_{-|j|,j}, \dots, xt^{|j|*}_{|j|,j})E_{2j}\big) = x\textstyle\sum_k t^{|j|*}_{k,j} t^{|j|}_{k,j} = x\, T^*_{2j}\, T_{2j} = x. \tag{3.25}$$

This shows the surjectivity of χ. Hence $\mathcal{C}(\mathrm{S}^2_q)^{|2j|+1}E_{2j} \cong M_{2j}$. □

In Proposition 3.1, we have shown that $L_N \cong T\times_{(u^N\sigma,1)}\mathbb{C}$. The next proposition presents explicit formulas for the projector matrices associated with the $\mathcal{C}(\mathrm{S}^2_q)$-modules $\mathcal{T}\times_{(u^N\sigma,1)}\mathbb{C}$.

Proposition 3.4. *Let $N \in \mathbb{N}$. Then $\mathcal{T}\times_{(u^N\sigma,1)}\mathbb{C} \cong \mathcal{C}(\mathrm{S}^2_q)^2 p_N$, where*

$$p_N = \begin{pmatrix} 1 & 0 \\ 0 & 1 - s^N s^{N*} \end{pmatrix}, \tag{3.26}$$

and $\mathcal{T}\times_{(u^{-N}\sigma,1)}\mathbb{C} \cong \mathcal{C}(\mathrm{S}^2_q)p_{-N}$, where

$$p_{-N} = s^N s^{N*}. \tag{3.27}$$

Proof. Recall that the formulas of Section 1.2 apply to our situation with the free rank 1 modules $E_1 = \mathcal{T}$, $E_2 = \mathbb{C}$ and $\pi_{1*}E_1 = \pi_{2*}E_2 = \mathcal{C}(\mathrm{S}^1)$. The isomorphism h in (1.9) is given by multiplication by $u^{\pm N}$, where $u = \mathrm{e}^{\mathrm{i}t} \in \mathcal{C}(\mathrm{S}^1)$. By the definition of the symbol map σ in (1.60), u^N and its inverse u^{-N} lift to s^N and s^{N*}, respectively. Inserting $c = s^N$ and $d = s^{N*}$ into Equation (1.12) and using $s^{N*}s^N = 1$, we obtain $\mathcal{T}\times_{(u^{-N}\sigma,1)}\mathbb{C} \cong (\mathcal{T}\times_{(\sigma,1)}\mathbb{C})^2 P_{-N}$ with

$$P_{-N} = \begin{pmatrix} (s^N s^{N*}, 1) & (0,0) \\ (0,0) & (0,0) \end{pmatrix}, \tag{3.28}$$

which is equivalent to $\mathcal{T}\times_{(u^{-N}\sigma,1)}\mathbb{C} \cong (\mathcal{T}\times_{(\sigma,1)}\mathbb{C})(s^N s^{N*}, 1)$. Applying the isomorphism ψ of Equation (3.12) to the last relation proves (3.27).

Analogously, inserting $c = s^{N*}$ and $d = s^N$ into (1.12) gives $\mathcal{T}\times_{(u^N\sigma,1)}\mathbb{C} \cong (\mathcal{T}\times_{(\sigma,1)}\mathbb{C})^2 P_N$, where

$$P_N = \begin{pmatrix} (1,1) & (0,0) \\ (0,0) & (1 - s^N s^{N*}, 0) \end{pmatrix}, \tag{3.29}$$

and applying the isomorphism ψ of (3.12) yields the result. □

Corollary 3.5. *For all $N \in \mathbb{Z}$, $[E_N] = [p_{-N}]$ in $\mathrm{K}_0(\mathcal{C}(\mathrm{S}^2_q))$.*

Proof. Since $L_N \cong T\times_{(u^N\sigma,1)}\mathbb{C}$, the claim follows from Proposition 3.4 and Corollary 3.3. □

3.3 Index pairing

Let A be a C^*-algebra, $p \in \mathrm{Mat}_n(A)$ a projection, and ρ_+ and ρ_- *-representations of A on a Hilbert space $\mathcal{H}$ such that $[(\rho_+, \rho_-)] \in \mathrm{K}^0(A)$. If the following trace exists, then the formula

$$\langle [(\rho_+, \rho_-)], [p] \rangle = \mathrm{Tr}_{\mathcal{H}}(\mathrm{Tr}_{\mathrm{Mat}_n}(\rho_+ - \rho_-)(p)) \tag{3.30}$$

yields a pairing between $\mathrm{K}^0(A)$ and $\mathrm{K}_0(A)$.

In this section, we compute the pairing between the K_0-classes of the projective $\mathcal{C}(\mathrm{S}^2_q)$-modules describing quantum line bundles and the two generators of $\mathrm{K}^0(A)$. By Corollary 3.3 and Corollary 3.5, we can take the projections p_N as representatives. Their simple form makes them very tractable for the calculation of the index pairing.

Proposition 3.6. *For all $N \in \mathbb{Z}$,*

$$\langle [(\varepsilon, \varepsilon_0)], [p_N] \rangle = 1, \qquad \langle [(\mathrm{id}, \varepsilon)], [p_N] \rangle = N, \tag{3.31}$$

where $[(\mathrm{id}, \varepsilon)]$ and $[(\varepsilon, \varepsilon_0)]$ denote the generators of $\mathrm{K}^0(\mathcal{C}(\mathrm{S}^2_q))$ given in Section 1.4.

Proof. Let $N \leq 0$. Then $p_N = s^{|N|}s^{|N|*} = (s^{|N|}s^{|N|*} - 1) + 1$, so that $\varepsilon(p_N) = 1$ and $\varepsilon_0(p_N) = ss^*$. Note that, for $n \in \mathbb{N}\backslash\{0\}$, $1 - s^n s^{n*}$ is the projection onto the subspace $\mathrm{span}\{e_0, \ldots, e_{n-1}\}$ of $\ell_2(\mathbb{N})$. In particular, it is of trace class. Therefore, by (3.30),

$$\langle [(\varepsilon, \varepsilon_0)], [p_N] \rangle = \mathrm{Tr}_{\ell_2(\mathbb{N})}(\varepsilon - \varepsilon_0)(p_N) = \mathrm{Tr}_{\ell_2(\mathbb{N})}(1 - ss^*) = 1, \tag{3.32}$$

$$\langle [(\mathrm{id}, \varepsilon)], [p_N] \rangle = \mathrm{Tr}_{\ell_2(\mathbb{N})}(\mathrm{id} - \varepsilon)(p_N) = \mathrm{Tr}_{\ell_2(\mathbb{N})}(s^{|N|}s^{|N|*} - 1) = N. \tag{3.33}$$

For $N > 0$, we have $\mathrm{Tr}_{\mathrm{Mat}_2}(p_N) = 2 - s^N s^{N*} = 2 - p_{-N}$. Clearly, $(\varepsilon - \varepsilon_0)(2) = 2(1 - ss^*)$ and $(\mathrm{id} - \varepsilon)(2) = 0$. Inserting these relations into (3.30) and using above calculations gives

$$\langle [(\varepsilon, \varepsilon_0)], [p_N] \rangle = \mathrm{Tr}_{\ell_2(\mathbb{N})}(\varepsilon - \varepsilon_0)(2 - p_{-N}) = \mathrm{Tr}_{\ell_2(\mathbb{N})}(1 - ss^*) = 1, \tag{3.34}$$

$$\langle [(\mathrm{id}, \varepsilon)], [p_N] \rangle = \mathrm{Tr}_{\ell_2(\mathbb{N})}(\mathrm{id} - \varepsilon)(2 - p_{-N}) = \mathrm{Tr}_{\ell_2(\mathbb{N})}(1 - s^N s^{N*}) = N, \tag{3.35}$$

which completes the proof. □

The above results are in agreement with the classical situation: The pairing $\langle [(\varepsilon, \varepsilon_0)], [p_N] \rangle$ yields the rank of the projective module $\mathcal{T} \times_{(u^{\pm N}\sigma, 1)} \mathbb{C}$, and $\langle [(\mathrm{id}, \varepsilon)], [p_N] \rangle$ computes the winding number, that is, the number of times that the map $u^N : \mathrm{S}^1 \to \mathrm{S}^1$ winds around the circle S^1.

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